INTRODUCTION TO THE
Theory of Equations

By

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PREFACE

In this book students are guided slowly through the proofs of the important general theorems in the elementary theory of algebraic equations. A background of plane trigonometry, plane analytic geometry, and the differential calculus is presupposed.

Development from the particular to the general is an outstanding feature of this book. For example, determinants of order three, determinants of order four, and determinants of order five are defined in such a way that these definitions illustrate all the details in the intricate general definition of determinants of order n which follows. Each property of determinants is proved completely if n is four or five, precisely as the general theorem is later proved. The same plan is used in the proofs of the theorems on systems of linear equations in n variables.

Attention is called also to the detailed exposition in this book. One type of amplification is separation of a complicated proof into simpler parts, as in the proof of Sturm's theorem and the illustrative material which precedes this proof. Again, clarifying restatement occurs frequently, as in the proof of the theorem characterizing the roots of the quartic equation by properties of its discriminant. Equations and theorems are also cited, as in the proof of the algebraic solution of the reduced cubic equation.

Numerous problems are inserted at appropriate intervals. In general, the odd-numbered problems constitute a complete set. The even-numbered problems may be used as an alternate set. Some problems illustrate proofs in the text.

The discussion of complex numbers in chapter 8 is independent of the preceding chapters. However, in my experience, a systematic study of the complete and precise statements and proofs of the general theorems in this book may well precede a study of the abstractions of modern algebra.

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BINOMIAL EQUATIONS

1. Functions and equations. The idea of function has proved to be of importance in mathematics and its applications. If z may assume several values, then $z^2 - 2z - 15$ is a function of z because each value of z determines a value of $z^2 - 2z - 15$. If z may assume several values, then z is called a variable. The statement that $f(z)$ is a single-valued function of z means that each value of z determines one value of $f(z)$. The symbol $f(c)$ denotes the value of $f(z)$ when z has the value c. Thus, if $f(z)$ is the particular function $z^2 - 2z - 15$, then $f(4)$ is $4^2 - 2\cdot 4 - 15$, that is, -7. A function may depend on more than one variable. For example, if z_1 may assume several values and independently z_2 may assume several values, then $2z_1 - z_2$ is a function of z_1 and z_2 because each pair of values of z_1 and z_2 determines a value of $2z_1 - z_2$. The statement that $f(z_1, \cdots, z_n)$ *is a single-valued function of the n independent variables* $z_1, \cdots, z_n$ means that each set of values of $z_1, \cdots, z_n$ determines one value of $f(z_1, \cdots, z_n)$. The symbol $f(c_1, \cdots, c_n)$ denotes the value of $f(z_1, \cdots, z_n)$ when $z_1, \cdots, z_n$ have the values $c_1, \cdots, c_n$ respectively. Thus, if $f(z_1, z_2)$ is the particular function $2z_1 - z_2$, then $f(2, -1)$ is $2\cdot 2 - (-1)$, that is, 5.

Equations are notations for questions about functions. For example, is there a value of z for which the value of $z^2 - 2z - 15$ is 9? The equation $z^2 - 2z - 15 = 9$ states this question. Is there a value of z for which the value of $2z^2 - 3z - 17$ is the same as the value of $z^2 - 2z - 15$? The equation $2z^2 - 3z - 17 = z^2 - 2z - 15$ states this question. If k is a number, is there a value c of z such that $f(c)$ is k? The equation $f(z) = k$ states this question. If $g(z)$ is a function of z, is there a value d of z such that $f(d)$ is $g(d)$? The equation $f(z) = g(z)$ states this question. Is there a value of z for which the value of $z^2 - 2z - 15$ is -16 and at the same time the value of $z^2 + z - 4$ is -2? The simultaneous equations $z^2 - 2z - 15 = -16$ and $z^2 + z - 4 = -2$ state

this question Are there values of z_1 and z_2 such that $2z_1 - z_2$ has the value 3 and $-z_1 + z_2$ has the value -2? The simultaneous equations $2z_1 - z_2 = 3$ and $-z_1 + z_2 = -2$ state this question If more than one equation is under consideration the equations are said to form a system The equations which are discussed in the elementary theory of equations are formed by inserting the symbol $=$ between a function and a number or between two functions

The statement that a number c is a root of an equation $f(z) = k$ means that the number $f(c)$ is the number k Thus 6 is a root of $z^2 - 2z - 15 = 9$ because $6^2 - 2\ 6 - 15$ is 9 The notation $6^2 - 2\ 6 - 15 = 9$ expresses this relation between these numbers Again the notation $f(c) = k$ expresses the statement that the number $f(c)$ is the number k If the symbol $=$ is inserted between numbers the result may be called an equation but it must be carefully distinguished from the equations which involve functions The statement that d is a root of $f(z) = g(z)$ means that $f(d) \quad g(d)$ A root of a system of equations is a root of each equation in the system Two systems are equivalent if they have the same roots A root of an equation is said to satisfy the equation Finding the roots of an equation is called solving the equation

The statement that *a set of numbers* $c_1 \quad c_n$ *is a solution of* $f(z_1 \quad z_n) = k$ means that $f(c_1 \quad c_n) = k$ A solution of a system of equations in more than one variable is a solution of each equation in the system For example the set of numbers $1 \ -1$ is a solution of the system $2z_1 \quad z_2 = 3$ and $-z_1 + z_2 = -2$ Two systems are equivalent if they have the same solutions

Later when there can be no misunderstanding it will be said that z is a root of $f(z) = 0$ and that $z_1 \quad , z_n$ is a solution of $f(z_1 \quad z_n) = 0$

2 Illustrations of use of factorization of a function If the operations indicated in $(z - 5)(z + 3)$ are performed the function $z^2 - 2z - 15$ is obtained This is the meaning of the statement that $(z - 5)(z + 3)$ is identically equal to $z^2 - 2z - 15$ The notation $(z - 5)(z + 3) \equiv z^2 - 2z - 15$ states this fact It is especially to be noted that these same operations can be performed if z is replaced by any value c of z This implies that if z is replaced in the identity $(z - 5)(z + 3) \equiv z^2 - 2z - 15$ by any

value c of z, then an equality $(c - 5)(c + 3) = c^2 - 2c - 15$ between numbers is obtained.

In general, $f(z_1, \cdots, z_n) \equiv g(z_1, \cdots, z_n)$ means that the result which is obtained by performing the operations indicated in $f(z_1, \cdots, z_n)$ is the same as that obtained by performing the operations indicated in $g(z_1, \cdots, z_n)$.

The factored form $(z - 5)(z + 3)$ may be used instead of the expression $z^2 - 2z - 15$ in the determination of functional values. Thus, when z has the value 1, the function has the value $(1 - 5)(1 + 3)$, that is, -16. When z has the value -1, the function has the value $(-1 - 5)(-1 + 3)$, that is, -12. When z has the value 5, the function has the value $(5 - 5)(5 + 3)$, that is, $0 \cdot 8$, that is, 0. When z has the value -3, the function has the value $(-3 - 5)(-3 + 3)$, that is, $(-8) \cdot 0$, that is, 0.

The use of the factored form of the function in computing functional values indicates that the functional value is 0 if the value of one of the factors $z - 5$ or $z + 3$ is 0, and that, if each of these values is different from 0, then the functional value is different from 0. Thus 5 and -3 are the only roots of $z^2 - 2z - 15 = 0$. This illustrates the general fact that, if $f(z)$ can be factored, then all the roots of $f(z) = 0$ are found from the factored form of this equation.

The following solution of the equation

$$z^3 = 1 \tag{1}$$

is a more complicated illustration of this general fact. Clearly the roots of (1) are the roots of the equation $z^3 - 1 = 0$. Also, $z^3 - 1 \equiv (z - 1)(z^2 + z + 1)$. Therefore the roots of (1) are the roots of

$$z - 1 = 0, \tag{2}$$

and the roots of

$$z^2 + z + 1 = 0. \tag{3}$$

Since the factorization of the function on the left-hand side of (3) is not obvious, the roots of (3) are found by the quadratic formula. They are the numbers $(-1/2) + (\sqrt{3}/2)i$ and $(-1/2) - (\sqrt{3}/2)i$. Hence there are three roots of the equation $z^3 = 1$, namely, these two complex numbers and the number 1. That each of these numbers is a root of $z^3 = 1$ can be checked by direct

substitution That the factored form of the quadratic function on the left-hand side of (3) is $\{z - [(-1/2) + (\sqrt{3}/2)i]\}$ $\{z - [(-1/2) - (\sqrt{3}/2)i]\}$ can also be checked Hence these three roots are the only roots of z^3 1

The equation $z^4 = 1$ can also be solved by factorization Thus $z^4 - 1 = (z^2 - 1)(z^2 + 1) = (z - 1)(z + 1)(z^2 + 1)$ By the quadratic formula the roots of $z^2 + 1 = 0$ are i and $-i$ Therefore the roots of $z^4 = 1$ are 1 -1 i $-i$

The equation

$$z^5 = 1 \tag{4}$$

is more difficult In fact only one root of this equation can be found by obvious factorization Thus $z^5 - 1 \equiv (z - 1)(z^4 + z^3 + z^2 + z + 1)$ Hence the roots of (4) are the roots of

$$z - 1 = 0 \tag{5}$$

and the roots of

$$z^4 + z^3 + z^2 + z + 1 = 0 \tag{6}$$

There is no obvious factorization of the left-hand side of (6) Later it will be explained how fourth-degree equations are solved

A new method of solving (1) will be explained in section 5 In section 8 it will be proved that this method is applicable to (4) and to

$$z^n = 1 \tag{7}$$

if n is a positive integer In section 9 it will be proved that this method is also applicable to

$$z^n \quad c \tag{8}$$

if n is a positive integer and c is a non zero complex number Equations of the forms (7) and (8) are called *binomial equations* because they contain exactly two terms

PROBLEMS

1 Show that $z^2 + z - 6 \equiv (z - 2)(z + 3)$ Using the form $z^2 + z - 6$ of this function compute its values when z has the values 4 3 2 -1 -3 Check by using the factored form of this function What are the roots of $z^2 + z - 6 = 0$?

2 Show that $z^2 \quad z - 6 \equiv (z + 2)(z - 3)$ Proceed as in problem 1 if z has the values 4 3 2 -1 -2

3. Verify that $z^3 + 1 = (z + 1)(z^2 - z + 1)$. Solve $z^3 = -1$ by factorization and the quadratic formula.

4. Solve $z^6 = 1$ by factorization, using the results of problem 3 and the roots of (1) which were found previously.

3. Complex numbers in trigonometric form. Complex numbers in trigonometric form are used in the new method of solving binomial equations which was mentioned at the end of section 2. In general, multiplication and division of complex numbers are simplified by using these numbers in trigonometric form.

The process of expressing complex numbers in trigonometric form is clarified by establishing a rectangular coordinate system in a plane and associating each complex number with a point in the plane. Two perpendicular lines in the plane are selected as X-axis and Y-axis, and the same unit of measure is used.

Several particular complex numbers will be expressed in trigonometric form before the process is applied to an arbitrary complex number. The complex number $(-1/2) + (\sqrt{3}/2)i$, which was a root of (1), determines the point whose X-coordinate is $-1/2$ and whose Y-coordinate is $\sqrt{3}/2$. This is the point P in Figure 1. It is known that P lies on the terminal line of the angle 120° in standard position and that the length of the line from the origin to P is 1. By trigonometry $\sin 120° = \sqrt{3}/2$, and $\cos 120° = -1/2$. Therefore $(-1/2) + (\sqrt{3}/2)i = 1(\cos 120° + i \sin 120°)$. The right-hand side of this equation is called *a trigonometric form of the complex number* $(-1/2) + (\sqrt{3}/2)i$. However, P is also on the terminal line of the angle $-240°$, and $\sin(-240°) = \sqrt{3}/2$ and $\cos(-240°) = -1/2$. Therefore $(-1/2) + (\sqrt{3}/2)i = 1[\cos(-240°) + i \sin(-240°)]$. The right-hand side of this equation is also called a trigonometric form of $(-1/2) + (\sqrt{3}/2)i$. In general, if $r = 1$ and θ denotes any angle which is coterminal with 120°, then $(-1/2) + (\sqrt{3}/2)i = r(\cos\theta + i \sin\theta)$. Each of these expressions involving r and θ is called a trigonometric form of $(-1/2) + (\sqrt{3}/2)i$. Each of these angles is called *an amplitude* of this complex number, and r is called *the modulus* of this complex number.

The complex number $1 - i$ will now be expressed in trigonometric form. Since $1 - i = 1 + (-1)i$, the number $1 - i$ determines the point whose X-coordinate is 1 and whose Y-coordinate is -1. This is the point Q in Figure 2. It is known that Q lies on the terminal line of the angle 315° in standard position and

that the length of the line from the origin to Q is $\sqrt{2}$ Also $\sin 315° = -1/\sqrt{2}$ and $\cos 315°$ $1/\sqrt{2}$ Therefore $1 - i$ $\sqrt{2}(\cos 315° + i \sin 315°)$ In the same way it is proved that $1 - i = \sqrt{2}[\cos(-45°) + i \sin(-45°)]$ and that $1 - i = \sqrt{2}$ $[\cos(315° + 360°) + i \sin(315° + 360°)]$ In general if $r = \sqrt{2}$ and θ designates any angle which is coterminal with 315° then $1 - i - r(\cos\theta + i \sin\theta)$ Each of these angles is an amplitude of $1 - i$ and r is the modulus of $1 - i$

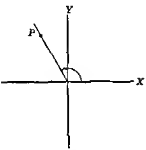

FIGURE 1

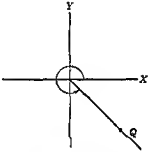

FIGURE 2

The complex number 1 will now be expressed in trigonometric form Since $1 = 1 + 0\,i$ the complex number 1 determines the point whose X coordinate is 1 and whose Y coordinate is 0 If $r = 1$ and θ is any angle which is coterminal with 0° then 1 $r(\cos\theta + i \sin\theta)$ Each of these angles is an amplitude of the complex number $1 + 0\,i$ and the modulus of this complex number is the positive number r

In each of these illustrations the rectangular coordinates of the point which is determined by the complex number are such that the value of r and a value of θ are known by experience It will now be explained how the value of r and a value of θ is determined by the literal non zero complex number $c + di$ Here c and d are real numbers The point S whose X coordinate is c and whose Y coordinate is d is determined by the complex number $c + di$ If r designates the length of the line from the origin to S and if θ designates any angle in standard position such that S lies on the terminal line of θ then by trigonometry

$$(9) \qquad \sin\theta = \frac{d}{r} \quad \text{and} \quad \cos\theta = \frac{c}{r}$$

Therefore

$$(10) \qquad c = r\cos\theta, \quad \text{and} \quad d = r\sin\theta.$$

From these expressions r is computed first. Thus, $c^2 + d^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$. Now r is positive because it is a length. Also it is a property of the real number system that there is one and only one positive number whose square is the positive number $c^2 + d^2$. *This positive number, whose square is $c^2 + d^2$, is designated by the symbol $\sqrt{c^2 + d^2}$.* Hence

$$(11) \qquad r = \sqrt{c^2 + d^2}.$$

If $c \neq 0$, a value of θ can be computed from the fact that $\tan\theta = d/c$ and the fact that the quadrant in which θ terminates is known. If $c = 0$ and if d is positive, then a value of θ is 90°. If $c = 0$ and if d is negative, then a value of θ is 270°. Then r and θ are known, and each of the expressions $r(\cos\theta + i\sin\theta)$ *is a trigonometric form of the complex number* $c + di$. The value of r which is found from (11) is *the modulus* of $c + di$, and each value of θ which is determined from (9) is *an amplitude* of $c + di$.

4. Multiplication and division of complex numbers in trigonometric form. The following lemma states the simple rule for multiplication of complex numbers in trigonometric form.

LEMMA 1. *The product of two complex numbers in trigonometric form is a complex number in trigonometric form. The modulus of the product is the product of the moduli of the factors. An amplitude of the product is the sum of an amplitude of the first factor and an amplitude of the second factor.*

PROOF. If $r(\cos\theta + i\sin\theta)$ and $s(\cos\phi + i\sin\phi)$ are two complex numbers in trigonometric form, then their product is $rs[(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)]$. From trigonometry it is known that

$$(12) \qquad \cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi,$$

$$(13) \qquad \sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi.$$

Hence the product of the two given complex numbers is $rs[\cos(\theta + \phi) + i\sin(\theta + \phi)]$.

The product $i(i+1)[(-1/2)+(\sqrt{3}/2)i]$ will be computed to illustrate the use of lemma 1. By (9) and (11) it is found that $1+i=\sqrt{2}(\cos 45^\circ + i\sin 45^\circ)$. Therefore by the lemma a trigonometric form of $(1+i)[(-1/2)+(\sqrt{3}/2)i]$ is $(\sqrt{2}\cdot 1)[\cos(45^\circ+120^\circ)+i\sin(45^\circ+120^\circ)]$. By (9) and (11) it is found that $0+1\cdot i=1(\cos 90^\circ+i\sin 90^\circ)$. Hence by the lemma applied to these last two complex numbers in trigonometric form, the modulus of $i(1+i)[(-1/2)+(\sqrt{3}/2)i]$ is $1(\sqrt{2}\cdot 1)$ and an amplitude is $90^\circ+(45^\circ+120^\circ)$. Hence a trigonometric form of the product $i(1+i)[(-1/2)+(\sqrt{3}/2)i]$ is

$$\sqrt{2}(\cos 255^\circ + i\sin 255^\circ) \tag{14}$$

The non-trigonometric form of the product $i(1+i)[(-1/2)+(\sqrt{3}/2)i]$ is obtained by multiplication in the usual manner. Thus $i(1+i)=i-1$. Also $(i-1)[(-1/2)+(\sqrt{3}/2)i]=[(1-\sqrt{3})/2]+[(-1-\sqrt{3})/2]i$. Therefore the usual form of the product $i(1+i)[(-1/2)+(\sqrt{3}/2)i]$ is

$$\left(\frac{1-\sqrt{3}}{2}\right)+\left(\frac{-1-\sqrt{3}}{2}\right)i \tag{15}$$

It will now be checked that the number (14) equals the number (15). Thus $\cos 255^\circ = \cos(180^\circ+75^\circ) = -\cos 75^\circ = -\cos(30^\circ+45^\circ) = -(\cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ) = -(\sqrt{3}/2)(\sqrt{2}/2)+(1/2)(\sqrt{2}/2)=(-\sqrt{3}+1)\sqrt{2}/4$. Similarly $\sin 255^\circ = -(1+\sqrt{3})\sqrt{2}/4$. If these results are used in (14), the number (15) is obtained.

It is to be noted that if the number of factors in a product is greater than three, then multiplication using trigonometric forms of the complex numbers is preferable.

PROBLEMS

1. Express each of the following complex numbers in trigonometric form: $-1+i$, $(-1/2)-(\sqrt{3}/2)i$, $-i$. Plot the point determined by each of these numbers.
2. Treat the numbers $-1-i$, $(1/2)-(\sqrt{3}/2)i$, i as in problem 1.
3. Find $(-1+i)[(-1/2)-(\sqrt{3}/2)i](-i)$ by the repeated use of lemma 1. Then perform this multiplication in non-trigonometric form. Show that the two results are equal.
4. Proceed as in problem 3 with $(-1-i)[(1/2)-(\sqrt{3}/2)i]i$.
5. By repeated use of lemma 1 find $[(-1/2)-(\sqrt{3}/2)i]^5$.

6. By repeated use of lemma 1 find $[(1/2) - (\sqrt{3}/2)i]^5$.

7. Prove that if $r(\cos\theta + i\sin\theta)$ is a trigonometric form of the complex number $c + di$, and if $c + di$ is not zero, then a trigonometric form of the complex number $1/(c + di)$ is

$$\frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]. \tag{16}$$

8. Prove that if $a + bi = r(\cos\theta + i\sin\theta)$, $c + di = s(\cos\phi + i\sin\phi)$, and $c + di$ is not zero, then $(a + bi)/(c + di) = (r/s)[\cos(\theta - \phi) + i\sin(\theta - \phi)]$. This is the *rule for division of complex numbers in trigonometric form*.

5. The cube roots of unity in trigonometric form. The new method of solving equation (1), which was mentioned in section 2, will now be explained. The desired root z is given the notation

$$z = r(\cos\theta + i\sin\theta). \tag{17}$$

By (9) and (11), a trigonometric form of the complex number 1, which appears on the right-hand side of the given equation (1), is $1(\cos 0° + i\sin 0°)$. Hence (1) becomes

$$[r(\cos\theta + i\sin\theta)]^3 = 1(\cos 0° + i\sin 0°). \tag{18}$$

Now by the lemma it is true that

$$[r(\cos\theta + i\sin\theta)]^2 = r^2(\cos 2\theta + i\sin 2\theta). \tag{19}$$

Hence, by the lemma applied to the right-hand side of (19) and $r(\cos\theta + i\sin\theta)$ as factors, it is true that

$$r(\cos\theta + i\sin\theta)\cdot[r(\cos\theta + i\sin\theta)]^2 = r^3(\cos 3\theta + i\sin 3\theta). \tag{20}$$

Hence (18) becomes

$$r^3(\cos 3\theta + i\sin 3\theta) = 1(\cos 0° + i\sin 0°). \tag{21}$$

Are there values of r and θ, that of r being positive, for which the complex number on the left-hand side of (21) is the complex number on the right-hand side? This is the question stated by (21). Therefore, by the definition of equality of complex numbers, there are two questions. First, is there a positive number r such that

$$r^3 = 1, \tag{22}$$

and, next, is there a value of θ such that 3θ is coterminal with $0°$? By the properties of the real number system there is one and only one positive number which satisfies (22). Therefore $r = 1$. Also

3θ is coterminal with $0°$ if and only if k is an integer such that

$$3\theta = 0° + k\,360° \tag{23}$$

Now when $k = 0$ then $\theta = 0°$ when $k = 1$ then $\theta = 120°$ when $k = 2$ then $\theta = 240°$ Hence three values of z are by (17)

$$\begin{aligned} z_0 &= 1(\cos 0° + i \sin 0°) \\ z_1 &= 1(\cos 120° + i \sin 120°) \\ z_2 &= 1(\cos 240° + i \sin 240°) \end{aligned} \tag{24}$$

These numbers z_0 z_1 z_2 are in trigonometric form Their ordinary forms are

$$z_0 = 1 \quad z_1 = \frac{-1}{2} + \frac{\sqrt{3}}{2}i \quad z_2 = \frac{-1}{2} - \frac{\sqrt{3}}{2}i \tag{25}$$

It is to be noted that these three roots of (1) which have been found using trigonometric forms are the values found from (2) and (3)

It will now be proved that if z_3 designates the value of z which is obtained if $k = 3$ in (23) then $z_3 = z_0$ If $k = 3$ then $\theta = (0° + 3\cdot 360°)/3 = 360°$ and $z_3 = 1(\cos 360° + i \sin 360°)$ However $360°$ is coterminal with $0°$ Therefore $z_3 = z_0$ In the same way it is proved that if z_4 designates the value of z which is obtained if $k = 4$ then $z_4 = z_1$ In general if two values of k differ by an integer which is divisible by 3 then the two values of θ which are determined from (23) differ by an integer which is divisible by $360°$ Therefore the roots of (1) which are determined by two such values of k have coterminal amplitudes and are equal Thus it has been proved that there are three and only three distinct numbers which satisfy the equation $z^3 = 1$ These numbers are the numbers (24) that is the numbers (25) These numbers are *the cube roots of unity*

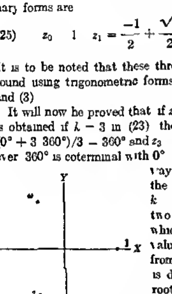

FIGURE 3

The symbol ω is also used to designate the complex number $(-1/2) + (\sqrt{3}/2)i$ which was designated by z_1 in (24) Therefore $\omega = 1(\cos 120° + i \sin 120°)$ Also $\omega^2 = 1(\cos 240° +$

$i \sin 240°$) by lemma 1. In this notation *the three cube roots of unity are 1, ω, ω^2.*

Each of the complex numbers 1, ω, ω^2 determines a point as in section 3. The notation P_1, P_ω, P_{ω^2} may be used for these points. In Figure 3 they are designated by the symbols 1, ω, ω^2. These points lie on the circle whose center is the origin and whose radius is unity. They are the vertices of an equilateral triangle.

6. De Moivre's theorem. This theorem will be used later in solving (4), (7), and (8) by the method mentioned at the end of section 2. De Moivre's theorem will be proved by mathematical induction. The first step in such a proof is verification that the theorem is true for at least one value of n. Usually verification is carried out for several small values of n. The second step is proof of the lemma for the induction.

DE MOIVRE'S THEOREM. *If n is a positive integer, then*

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta. \tag{26}$$

PROOF. De Moivre's theorem is true for the value 2 of n because by lemma 1

$$(\cos \theta + i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta. \tag{27}$$

If both sides of (27) are multiplied by $\cos \theta + i \sin \theta$, and if lemma 1 is applied to the right-hand side of the result, it is found that

$$(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta. \tag{28}$$

Therefore De Moivre's theorem is true for the value 3 of n.

LEMMA FOR THE INDUCTION. *If n_0 is a value of n for which (26) is true, then $n_0 + 1$ is a value of n for which (26) is true.*

By the statement of the lemma it is known that

$$(\cos \theta + i \sin \theta)^{n_0} = \cos n_0\theta + i \sin n_0\theta. \tag{29}$$

It is to be proved that

$$(\cos \theta + i \sin \theta)^{n_0+1} = \cos (n_0 + 1)\theta + i \sin (n_0 + 1)\theta. \tag{30}$$

If both sides of (29) are multiplied by $\cos \theta + i \sin \theta$, the result is

$$(\cos \theta + i \sin \theta) \cdot (\cos \theta + i \sin \theta)^{n_0} = (\cos \theta + i \sin \theta) \cdot (\cos n_0\theta + i \sin n_0\theta). \tag{31}$$

Now the left-hand side of (31) is the left-hand side of (30) Also, by lemma 1 the right-hand side of (31) is the right-hand side of (30) This completes the proof of the lemma for the induction

Since (26) has been verified for the value 3 of n it is known by the lemma that (26) is true for the value $3 + 1$, that is, the value 4 of n Again, since (26) is true for the value 4 of n, it is known by the lemma that (26) is true for the value $4 + 1$, that is the value 5 of n Therefore, by a continuation of this process, (26) is true for each positive integral value of n

7 The fifth roots of unity De Moivre's theorem will now be used to solve (4) If z is given the notation (17), then (4) becomes $[r(\cos\theta + i\sin\theta)]^5 = 1(\cos 0° + i\sin 0°)$ Hence by De Moivre's theorem (4) becomes

$$r^5(\cos 5\theta + i\sin 5\theta) = 1(\cos 0° + i\sin 0°) \tag{32}$$

Then as in the discussion following (21) r is determined by the fact that r is positive and the equation $r^5 = 1$ and θ is determined from

$$5\theta = 0° + k\ 360 \quad k \text{ an integer} \tag{33}$$

Therefore $r = 1$ Also the values 0 1 2 3 4 of k yield respectively the roots

$$\begin{aligned}
z_0 &= 1(\cos 0° + i\sin 0°),\\
z_1 &= 1\left(\cos\frac{1\ 360°}{5} + i\sin\frac{1\ 360°}{5}\right),\\
z_2 &= 1\left(\cos\frac{2\ 360°}{5} + i\sin\frac{2\ 360°}{5}\right),\\
z_3 &= 1\left(\cos\frac{3\ 360°}{5} + i\sin\frac{3\ 360°}{5}\right),\\
z_4 &= 1\left(\cos\frac{4\ 360°}{5} + i\sin\frac{4\ 360°}{5}\right),
\end{aligned} \tag{34}$$

of (4) If two values of k differ by an integer which is divisible by 5 then the two values θ which are determined by (33) differ

by an integer which is divisible by 360°. Therefore the two roots (17) of (4) which are determined by such values of k have coterminal amplitudes and are equal. Thus it has been proved that there are five and only five distinct numbers which are roots of (4). These numbers are the numbers (34). These numbers are *the fifth roots of unity.*

The symbol ϵ is often used to designate the number which was given the notation z_1 in (34). Therefore by De Moivre's theorem $\epsilon^2 = z_2$, $\epsilon^3 = z_3$, $\epsilon^4 = z_4$. In this notation the five fifth roots of unity are 1, ϵ, ϵ^2, ϵ^3, ϵ^4. These complex numbers determine respectively the five points which are designated by 1, ϵ, ϵ^2, ϵ^3, ϵ^4 in Figure 4. These points lie on the circle whose center is the origin and whose radius is unity. They are the vertices of a regular pentagon.

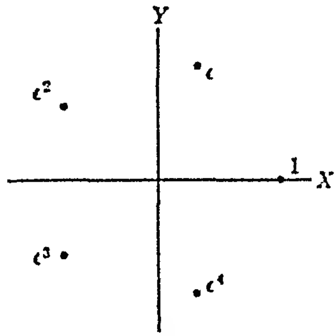

FIGURE 4.

8. The nth roots of unity. De Moivre's theorem will now be used to solve (7). If the notation (17) is used for z, then (7) becomes

(35) $$[r(\cos\theta + i\sin\theta)]^n = 1(\cos 0° + i\sin 0°).$$

Hence by De Moivre's theorem (7) becomes

(36) $$r^n(\cos n\theta + i\sin n\theta) = 1(\cos 0° + i\sin 0°).$$

Therefore r and θ are determined from

(37) $$n\theta = 0° + k\cdot 360°, \quad k \text{ an integer},$$

(38) $$r^n = 1.$$

Therefore $r = 1$. Also, if two values of k differ by an integer which is divisible by n, then the two values of θ determined from (37) differ by an integer which is divisible by 360°. Then the two roots (17) which are determined by such values of k have coterminal amplitudes and are equal. Thus it has been proved that there are n and only n distinct numbers which are roots of $z^n = 1$.

These numbers are

$$
\begin{aligned}
z_0 &= 1(\cos 0^\circ + i \sin 0^\circ),\\
z_1 &= 1\left(\cos \frac{1\ 360^\circ}{n} + i \sin \frac{1\ 360^\circ}{n}\right),\\
z_2 &= 1\left(\cos \frac{2\ 360^\circ}{n} + i \sin \frac{2\ 360^\circ}{n}\right), \qquad (39)\\
&\cdots\\
z_{n-1} &= 1\left[\cos \frac{(n-1)\ 360^\circ}{n} + i \sin \frac{(n-1)\ 360^\circ}{n}\right]
\end{aligned}
$$

These numbers are *the nth roots of unity*

These n numbers (39) may be written simultaneously by

$$
z_k = 1\left(\cos \frac{k\ 360^\circ}{n} + i \sin \frac{k\ 360^\circ}{n}\right), \qquad (40)
$$

$$
k = 0\ 1\ 2 \qquad n-1
$$

Now z_0 is actually a real number because its amplitude is 0°, and hence by (9) the Y coordinate of the point determined by z_0 is zero Also if n is an even integer then $n/2$ is an integer and the value $n/2$ of k gives $\theta = 360^\circ/2 = 180^\circ$ By (9) the Y coordinate of the point determined by $z_{n/2}$ is zero Hence if n is even, then $z_{n/2}$ is a real number If n is odd then z_0 is the only real number among the roots (40) These results are summarized in theorem 1

THEOREM 1 *If n is a positive integer then there are n distinct nth roots of unity They are the numbers (39) that is the numbers (40) If n is odd, then z_0 is the only real number among these roots If n is even, then z_0 and $z_{n/2}$ are the only real roots*

PROBLEMS

1 Apply theorem 1 to find the fourth roots of unity Show that these roots are the same as those found in section 2 by factorization

2 Apply theorem 1 to find the sixth roots of unity Show that these roots are the same as those found in problem 4 in section 2

3 By comparing the results of problem 2 with (24) determine which of the sixth roots of unity are cube roots of unity and which are not cube roots of unity

4. Apply theorem 1 to find the eighth roots of unity. By comparing these results with the results of problem 1 determine precisely which of the eighth roots of unity are fourth roots of unity and which are not fourth roots of unity.

If the symbol ϵ is used to designate the number which was given the notation z_1 in (39), then by De Moivre's theorem $\epsilon^2 = z_2$, $\cdots$, $\epsilon^{n-1} = z_{n-1}$. In this notation the n nth roots of unity are $1, \epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$. These complex numbers determine respectively n points which lie on the circle whose center is the origin and whose radius is unity. They are the vertices of a regular polygon of n sides. The following theorem has been proved.

THEOREM 2. *If n is a positive integer, and if ϵ designates the complex number $\cos(360°/n) + i \sin(360°/n)$, then the n nth roots of unity are $1, \epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$.*

If $n = 3$, then ϵ in theorem 2 is the complex number which was designated by ω at the end of section 5.

9. The nth roots of an arbitrary non-zero complex number. De Moivre's theorem will now be used to solve the particular binomial equation

$$z^4 = \omega. \tag{41}$$

If the desired root is given the notation (17), then (41) becomes

$$[r(\cos\theta + i\sin\theta)]^4 = 1(\cos 120° + i\sin 120°). \tag{42}$$

Hence by De Moivre's theorem (41) becomes

$$r^4(\cos 4\theta + i\sin 4\theta) = 1(\cos 120° + i\sin 120°). \tag{43}$$

Hence $r = 1$, and θ is determined from

$$4\theta = 120° + k\cdot 360°, \quad k \text{ an integer}. \tag{44}$$

Hence

$$\theta = 30° + k\cdot 90°, \quad k \text{ an integer}. \tag{45}$$

Now, if two values of k differ by an integer which is divisible by 4, then the values of θ differ by an integer which is divisible by 360°. Hence there are four and only four distinct roots of (41). They are

$$z_k = 1[\cos(30° + k\cdot 90°) + i\sin(30° + k\cdot 90°)], \tag{46}$$
$$k = 0, 1, 2, 3.$$

It is to be noted especially that each equation of the type (7) has on the right-hand side a number whose modulus is 1 Equation (41) also has this property

As an illustration of the solution of an equation of the type (8) in which the modulus of c is not 1, the equation

$$z^3 = 1 + i \tag{47}$$

will now be solved If z is given the notation (17), then (47) becomes

$$[r(\cos\theta + i\sin\theta)]^3 = \sqrt{2}(\cos 45^\circ + i\sin 45^\circ) \tag{48}$$

Hence by De Moivre's theorem (47) becomes

$$r^3(\cos 3\theta + i\sin 3\theta) = \sqrt{2}(\cos 45^\circ + i\sin 45^\circ) \tag{49}$$

Therefore r and θ are determined from

$$r^3 = \sqrt{2} \tag{50}$$

$$3\theta = 45^\circ + k\ 360^\circ \quad k \text{ an integer,} \tag{51}$$

and the fact that r is a positive number It is a property of the real number system that there is one and only one positive number which satisfies (50) This number is $\sqrt[3]{\sqrt{2}}$ that is $\sqrt[6]{2}$ Also the values 0 1 2 of k are the only values of k such that the values of θ which are determined from them by (51) are noncoterminal Hence there are three and only three distinct roots of (47) They are the numbers

$$\begin{aligned} z_0 &= \sqrt[6]{2}[\cos(15^\circ + 0\ 120^\circ) + i\sin(15^\circ + 0\ 120^\circ)], \\ z_1 &= \sqrt[6]{2}[\cos(15^\circ + 1\ 120^\circ) + i\sin(15^\circ + 1\ 120^\circ)], \\ z_2 &= \sqrt[6]{2}[\cos(15^\circ + 2\ 120^\circ) + i\sin(15^\circ + 2\ 120^\circ)] \end{aligned} \tag{52}$$

No new ideas are involved in the solution of the general binomial equation (8) If the required root z is given the notation $r(\cos\theta + i\sin\theta)$ and the complex number c the notation $s(\cos\alpha + i\sin\alpha)$, then by De Moivre's theorem the equation $z^n = c$ becomes

$$r^n(\cos n\theta + i\sin n\theta) = s(\cos\alpha + i\sin\alpha) \tag{53}$$

Therefore r and θ are determined from

$$r^n = s \quad r \text{ positive,} \tag{54}$$

$$n\theta = \alpha + k\ 360^\circ, \quad k \text{ an integer} \tag{55}$$

Now s is a positive number. Also it is a property of the real number system that there is one and only one positive number whose nth power is s. This positive number is designated by $\sqrt[n]{s}$. Therefore $r = \sqrt[n]{s}$.

Also (55) becomes

(56) $$\theta = \frac{\alpha + k\cdot 360^\circ}{n}, \quad k \text{ an integer.}$$

Now, if two values of k differ by an integer which is a multiple of n, then the two values of θ determined from (56) differ by an integer which is a multiple of 360°. Therefore the values of the root $r(\cos\theta + i\sin\theta)$ which are determined by these two values of k are equal. Hence there are exactly n distinct roots of the equation $z^n = c$. They are the numbers $z_0, z_1, \cdots, z_{n-1}$, whose values are

(57) $$\begin{aligned} z_0 &= \sqrt[n]{s}\left(\cos\frac{\alpha}{n} + i\sin\frac{\alpha}{n}\right), \\ z_1 &= \sqrt[n]{s}\left(\cos\frac{\alpha + 1\cdot 360^\circ}{n} + i\sin\frac{\alpha + 1\cdot 360^\circ}{n}\right), \\ &\cdot\qquad\cdot\qquad\cdot \\ &\cdot\qquad\cdot\qquad\cdot \\ &\cdot\qquad\cdot\qquad\cdot \\ z_{n-1} &= \sqrt[n]{s}\left[\cos\frac{\alpha + (n-1)\cdot 360^\circ}{n}\right. \\ &\qquad\qquad \left. + i\sin\frac{\alpha + (n-1)\cdot 360^\circ}{n}\right]. \end{aligned}$$

If ϵ is defined as in theorem 2, then the product $z_0\cdot\epsilon$ is the number z_1. In general,

(58) $$z_0\epsilon^i = z_i \quad (i = 1, 2, \cdots, n-1).$$

Hence the following theorem has been proved.

THEOREM 3. *Let n be a positive integer and ϵ designate the complex number $\cos(360^\circ/n) + i\sin(360^\circ/n)$. Let a trigonometric form of the non-zero complex number c be $s(\cos\alpha + i\sin\alpha)$. Then there are exactly n roots of the equation $z^n = c$. They are the complex number $\sqrt[n]{s}[\cos(\alpha/n) + i\sin(\alpha/n)]$ and the products of this complex number by $\epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$.*

If $c > 0$ and n is odd, then z_0 is the only real root of (8) If $c > 0$ and n is even, then z_0 and $z_0\epsilon^{n/2}$ are the only real roots If $c < 0$ and n is odd, then $z_{(n-1)/2}$ is the only real root If $c < 0$ and n is even, there are no real roots

COROLLARY 1 *If d is a non-zero real number, then the cube roots of d^3 are d, $d\omega$, $d\omega^2$*

PROOF *If $d > 0$ and if $n = 3$, $s = d$, $\alpha = 0°$ in theorem 3, then* the result stated in the corollary is obtained If $d < 0$, then $-d > 0$ By the preceding case the cube roots of $(-d)^3$ are $-d$, $-d\omega$, $-d\omega^2$ Therefore the result stated in the corollary is obtained by multiplication by -1

COROLLARY 2 *If d and b are real numbers such that $d^3 = b^3$, then $d = b$*

PROOF If d or b is 0, then the other is 0, and they are equal If $d > 0$ then $d = b$ by the property of the real number system which follows (55) If $d < 0$ then $-d > 0$ and by the preceding case $-d = -b$ Therefore $d = b$

PROBLEMS

1 Find the cube roots of ω and the fifth roots of $-i$
2 Find the cube roots of $-\omega$ and the fifth roots of i
3 Find the fifth roots of -1 and the fifth roots of ω
4 Find the sixth roots of -1 and the sixth roots of $-\omega$
5 Find the fourth roots of $-1 + i$ and the fifth roots of $(1/2) - (\sqrt{3}/2)i$
6 Find the fourth roots of $-1 - i$ and the fifth roots of $(-1/2) - (\sqrt{3}/2)i$

10 Relation between the cube roots of a complex number and the cube roots of the conjugate complex number If $c + di$ is a complex number, then *the conjugate complex number* is, by definition, $c - di$ By (11) these complex numbers have the same positive number as moduli This modulus will be designated by s It is also true, by (9) and by trigonometry, that, if α is an amplitude of $c + di$, then $-\alpha$ is an amplitude of $c - di$ Hence

$$c + di = s(\cos\alpha + i\sin\alpha) \tag{59}$$

$$c - di = s[\cos(-\alpha) + i\sin(-\alpha)] \tag{60}$$

The three cube roots of $c + di$ will be designated by A_1, A_2, A_3. Therefore, by theorem 3 and (59),

$$A_1 = \sqrt[3]{s}\left(\cos\frac{\alpha}{3} + i\sin\frac{\alpha}{3}\right),$$

(61) $$A_2 = A_1\omega = \sqrt[3]{s}\left(\cos\frac{\alpha + 360°}{3} + i\sin\frac{\alpha + 360°}{3}\right),$$

$$A_3 = A_1\omega^2 = \sqrt[3]{s}\left(\cos\frac{\alpha + 2\cdot 360°}{3} + i\sin\frac{\alpha + 2\cdot 360°}{3}\right).$$

The three cube roots of $c - di$ will be designated by B_1, B_2, B_3. Therefore, by theorem 3 and (60),

$$B_1 = \sqrt[3]{s}\left(\cos\frac{-\alpha}{3} + i\sin\frac{-\alpha}{3}\right),$$

(62) $$B_2 = B_1\omega = \sqrt[3]{s}\left(\cos\frac{-\alpha + 360°}{3} + i\sin\frac{-\alpha + 360°}{3}\right),$$

$$B_3 = B_1\omega^2 = \sqrt[3]{s}\left(\cos\frac{-\alpha+2\cdot 360°}{3} + i\sin\frac{-\alpha+2\cdot 360°}{3}\right).$$

By lemma 1 it is true that

(63) $$A_1B_1 = (\sqrt[3]{s}\,)^2(\cos 0° + i\sin 0°).$$

Hence A_1B_1 is real. Also, by (61_1) and (62_1), it is true that the modulus of B_1 is the modulus of A_1 and that an amplitude of B_1 is the negative of an amplitude of A_1. Hence B_1 is the conjugate of A_1. Again, by (61_2) and (62_3), it is true that $A_2B_3 = A_1\omega B_1\omega^2 = A_1B_1\omega^3 = A_1B_1$. Now the angle $(-\alpha + 2\cdot 360°)/3$, which is an amplitude of B_3, is not the negative of the angle $(\alpha + 360°)/3$, which is an amplitude of A_2. But $(-\alpha + 2\cdot 360°)/3$ is coterminal with $-(\alpha + 360°)/3$, since their difference, $[(-\alpha + 2\cdot 360°)/3] - [-(\alpha + 360°)/3]$, is an integer which is a multiple of 360°. Hence $-(\alpha + 360°)/3$ is an amplitude of B_3. Hence B_3 is the conjugate of A_2. Similarly it is proved that A_3B_2 is real and B_2 is the conjugate of A_3. In lemma 2 these results are summarized.

LEMMA 2. *Let a trigonometric form of the complex number $c + di$ be $s(\cos\alpha + i\sin\alpha)$. Then a trigonometric form of $c - di$ is*

$s[\cos(-\alpha) + i \sin(-\alpha)]$ Let A_1 designate the particular cube root $\sqrt[3]{s}[\cos(\alpha/3) + i \sin(\alpha/3)]$ of $c + di$ Let B_1 designate the particular cube root $\sqrt[3]{s}[\cos(-\alpha)/3 + i \sin(-\alpha)/3]$ of $c - di$ Then the three cube roots of $c + di$ are A_1 ωA_1 $\omega^2 A_1$ and the three cube roots of $c - di$ are B_1 ωB_1 $\omega^2 B_1$ Also B_1 is the conjugate of A_1 and A_1 B_1 is a real number Also $\omega^2 B_1$ is the conjugate of ωA_1 and ωA_1 $\omega^2 B_1$ is a real number Also ωB_1 is the conjugate of $\omega^2 A_1$ and $\omega^2 A_1$ ωB_1 is a real number

Lemma 2 states the important fact that the three cube roots of a complex number and the three cube roots of the conjugate complex number can be paired so that in each pair the two numbers are conjugate complex numbers and their product is a real number

It is to be noted especially that the proof of lemma 2 holds regardless of whether d in (59) is or is not zero Therefore lemma 2 is true even if $c + di$ is a real number

PROBLEMS

1 Find the cube roots of $-1 + i$ and the cube roots of $-1 - i$ Show that these cube roots can be paired as indicated in lemma 2.

2 Proceed as in problem 1 for ω and its conjugate

3 Proceed as in problem 1 for 8 and its conjugate

4 Proceed as in problem 1 for -8 and its conjugate

5 Proceed as in problem 1 for $(1/2) + (\sqrt{3}/2)i$ and its conjugate

6 Proceed as in problem 1 for $1 - i$ and its conjugate

There is further discussion of complex numbers in chapter 8 Other interesting and important facts about roots of unity are discussed in the references cited at the end of this book

CHAPTER 2

CUBIC AND QUARTIC EQUATIONS

1. The general cubic equation and its reduced cubic equation. In section 2 of chapter 1 the equation $x^3 = 1$ was solved by factoring the function $x^3 - 1$. The equation $2x^3 - 5x^2 - 4x + 3 = 0$ can also be solved by factoring, since $2x^3 - 5x^2 - 4x + 3 \equiv (x + 1)(x - 3)(2x - 1)$. A method of finding such simple factors of functions of this kind will be explained in chapter 3. However, there are cubic functions which have no simple factors. Therefore a general method of solving cubic equations will now be explained.

It is assumed that the coefficients a, b, c, d of the general cubic equation

$$ax^3 + bx^2 + cx + d = 0 \tag{1}$$

are real numbers, and that the leading coefficient a is not zero. A *real cubic equation* is an equation of the form (1) whose coefficients have these two properties.

A number k is a root of (1) if and only if it is a root of

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0. \tag{2}$$

This is the meaning of the statement that (1) and (2) are equivalent equations.

It will now be explained how the roots of (2) can be found from the roots of a more simple cubic equation. If x and y are related by the equation

$$x = y - \frac{b}{3a}, \tag{3}$$

then x determines y and y determines x. If x is a root of (2) and if y is computed by (3), then y is a root of the equation which is obtained by substituting from (3) in (2). This equation is

$$y^3 + \left(\frac{c}{a} - \frac{b^2}{3a^2}\right)y + \left(\frac{d}{a} - \frac{bc}{3a^2} + \frac{2b^3}{27a^3}\right) = 0. \tag{4}$$

Conversely if y is a root of (4) and if (3) is used to compute x then x is a root of (2) The complicated coefficients in (4) enter frequently in the following proof Hence they will be abbreviated by

$$(5) \qquad C = \frac{c}{a} - \frac{b^2}{3a^2} \qquad D = \frac{d}{a} - \frac{bc}{3a^2} + \frac{2b^3}{27a^3}$$

Then (4) becomes

$$(6) \qquad y^3 + Cy + D = 0$$

Therefore all the roots of (1) are obtained by solving the more simple equation (6) for y and then determining x by (3) Equation (6) determined from (1) by (5) is called *the reduced cubic equation for (1)*

2 Algebraic solution of the reduced cubic equation If $C = 0$ then the equation (6) is the binomial equation

$$(7) \qquad y^3 = -D$$

A method of solving this equation was explained in section 9 of chapter 1 A method of solving (6) if $C \neq 0$ will now be explained

If $C \neq 0$ then each value of y in $3z^2 - 3yz - C = 0$ determines two non zero values of z Conversely each non zero value of z determines one value of y because then this relation can be written in the form

$$(8) \qquad y = z - \frac{C}{3z}$$

In particular a value of y which satisfies (6) determines two non zero values of z which satisfy the equation obtained by substituting (8) in (6) This equation is

$$(9) \qquad z^3 - \frac{C^3}{27z^3} + D = 0 \quad C \neq 0$$

Conversely each non zero value of z which satisfies (9) determines a value of y which satisfies (6) The non zero values of z which satisfy (9) are the values of z which satisfy

$$(10) \qquad z^6 + Dz^3 - \frac{C^3}{27} = 0 \quad C \neq 0$$

Therefore, if $C \neq 0$, all roots y of (6) are found by using all roots z of (10) in (8).

Now (10) is a quadratic equation in z^3. The discriminant of this quadratic equation will be designated by R. Hence

$$R = D^2 + 4\frac{C^3}{27}. \tag{11}$$

Also, the two numbers which are the roots of this quadratic equation are $(-D + \sqrt{R})/2$ and $(-D - \sqrt{R})/2$. Hence, if z satisfies (10), then z satisfies one of the equations

$$z^3 = \frac{-D + \sqrt{R}}{2}, \tag{12}$$

$$z^3 = \frac{-D - \sqrt{R}}{2}. \tag{13}$$

Conversely, if z satisfies (12) or (13), then z satisfies (10).

It was proved in section 9 of chapter 1 that there are exactly three roots of (12) and exactly three roots of (13). The three roots of (12) will be designated by z_1, z_2, z_3 and the three roots of (13) by z_4, z_5, z_6. Then, by (8), there are six values of y:

$$y_i = z_i - \frac{C}{3z_i} \quad (i = 1, \cdots, 6). \tag{14}$$

It will now be proved that

$$y_4 = y_1, \quad y_5 = y_3, \quad y_6 = y_2. \tag{15}$$

Hence it will follow that there are exactly three roots of (6) if $C \neq 0$.

The first step in the proof of (15) is the proof that

$$z_1z_4 = \frac{-C}{3}, \quad z_2z_6 = \frac{-C}{3}, \quad z_3z_5 = \frac{-C}{3}. \tag{16}$$

The fact that $z_1z_4 = -C/3$ will be established by using corollary 2 of chapter 1. Thus, first it will be proved that z_1z_4 is real and that $-C/3$ is real. Then it will be proved that $(z_1z_4)^3 = (-C/3)^3$. Now, if $R \leq 0$, then the right-hand sides of (12) and (13) are indeed conjugate complex numbers and therefore, by lemma 2 of

chapter 1, it is true that z_1z_4 is real But, if $R > 0$, then $(-D + \sqrt{R})/2$ and $(-D - \sqrt{R})/2$ are unequal real numbers, and hence lemma 2 cannot be used But then, by corollary 1 of chapter 1, z_1 is real and z_4 is real, and hence z_1z_4 is real Next, by the hypothesis that a, b, c, d in (1) are real, it is true by (5) that $-C/3$ is real Finally, $z_1^3 = (-D + \sqrt{R})/2$, since z_1 is a root of (12), also $z_4^3 = (-D - \sqrt{R})/2$, since z_4 is a root of (13) Hence $z_1^3z_4^3 = (D^2 - R)/4$ Hence, by (11), it is true that $(z_1z_4)^3 = (-C/3)^3$ This proves the first equality in (16) The other equalities in (16) are proved in the same way

The first equality in (15) will now be proved Thus, by (14), $y_4 = z_4 - (C/3z_4)$ Hence, by (16), $y_4 = z_4 + z_1$ Again, by (14) $y_1 = z_1 - (C/3z_1)$, and hence, by (16), $y_1 = z_1 + z_4$ Therefore $y_4 = y_1$ The other equalities in (15) are proved in the same way It is especially to be noted that

$$y_1 = z_1 + z_4, \quad y_2 = z_2 + z_6, \quad y_3 = z_3 + z_5$$

Also, by lemma 2 of chapter 1, $z_2 = \omega z_1$, $z_3 = \omega^2 z_1$, and $z_5 = \omega z_4$, $z_6 = \omega^2 z_4$ Therefore

(17) $\quad y_1 = z_1 + z_4, \quad y_2 = \omega z_1 + \omega^2 z_4, \quad y_3 = \omega^2 z_1 + \omega z_4$

This completes the proof of theorem 1

THEOREM 1 *The general cubic equation $ax^3 + bx^2 + cx + d = 0$ has exactly three roots These roots are the numbers $y_1 - (b/3a)$, $y_2 - (b/3a)$, $y_3 - (b/3a)$, in which y_1, y_2, y_3 are the three roots of the reduced equation $y^3 + Cy + D = 0$, whose coefficients C and D are determined from the coefficients a, b, c, d by (5) If $C = 0$, then the roots y_1, y_2, y_3 are found by theorem 3 of chapter 1 If $C \neq 0$, then the number R is determined from C and D by (11) Then the roots y_1, y_2, y_3 are found from the roots of two auxiliary equations (12) and (13) If z_1 is one of the roots of (12), then there is exactly one root of (13), designated by z_4 such that the product z_1z_4 is real Then the three roots y_1, y_2, y_3 are given by (17)*

It is to be noted especially that only addition, subtraction, multiplication, division, extraction of roots are used to express z_1 and z_4 in terms of the coefficients C and D These processes are *the algebraic processes* Therefore these formulas give an *algebraic solution of the cubic equation* These expressions are known as *Cardan's formulas*

PROBLEMS

Solve the following equations by Cardan's formulas.

1. $8x^3 + 24x^2 + 48x - 31 = 0.$
2. $x^3 + 6x^2 + 18x - (181/27) = 0.$
3. $x^3 - 6x^2 + 14x - (343/27) = 0.$
4. $x^3 - 3x^2 + 4x - (28/27) = 0.$
5. $27x^3 - 27x^2 + 117x - 59 = 0.$
6. $8x^3 + 12x^2 - 18x + 9 = 0.$
7. $3x^3 + 18x^2 + 27x - 4 = 0.$
8. $27x^3 + 27x^2 + 144x - 64 = 0.$
9. $x^3 + 3x^2 + 5x + (100/27) = 0.$
10. $x^3 + 3x^2 + 2x - (28/27) = 0.$
11. $x^3 - 3x^2 + 4x - (1/54) = 0.$
12. $3x^3 - 3x^2 - 2x + (268/27) = 0.$
13. $x^3 - 6x^2 + 15x - 19 = 0.$
14. $x^3 + 6x^2 + 15x + 11 = 0.$
15. $x^3 + 6x^2 + 9x + 6 = 0.$
16. $x^3 - 6x^2 + 9x + 4 = 0.$

3. Trigonometric solution of the cubic equation with real roots. It is to be noted that in each problem of the preceding list the numbers on the right-hand sides of (12) and (13) are real numbers, because R is a positive number. Thus in these problems only the cube roots of real numbers are needed. If a numerical cubic should lead to a value of R which is a negative number, then the cube roots of two complex numbers which are not real would be needed. They could be found by theorem 3 of chapter 1. However, there is a more practical method of solving a cubic for which $R < 0$. This method will now be explained.

In the equation

$$y^3 + Cy + D = 0 \tag{18}$$

it is now assumed that $C \neq 0$ and that

$$D^2 + 4\frac{C^3}{27} < 0. \tag{19}$$

Also D is a real number, by (5) and the hypothesis that the coefficients a, b, c, d of (1) are real numbers. Hence $D^2 \geqq 0$, and $C < 0$ by (19).

If $D = 0$, then equation (18) becomes

$$y^3 + Cy = 0. \tag{20}$$

Its roots are the real numbers $0, +\sqrt{-C}, -\sqrt{-C}$.

If $D \neq 0$, the following method involves only real numbers and is preferable to Cardan's method for obtaining numerical results. Since $C < 0$, the number $\sqrt{-4C/3}$ is positive. Then each value of y in

$$y = \sqrt{\frac{-4C}{3}}\,z \tag{21}$$

determines a value of z Conversely each value of z determines a value of y In particular a value of y which satisfies (18) determines a value of z which satisfies the equation obtained by substituting (21) in (18) This equation is

$$\left(\sqrt{\frac{-4C}{3}}\right)^3 z^3 + C\sqrt{\frac{-4C}{3}}\,z + D = 0 \tag{22}$$

Conversely a value of z which satisfies (22) determines a value of y which satisfies (18) Now the roots of (22) are the roots of

$$z^3 - \frac{3}{4}z + \frac{D}{(\sqrt{-4C/3})^3} = 0 \tag{23}$$

Therefore the roots y of (18) are found by using the roots z of (23) in (21)

It will now be proved that if α is any angle the roots of

$$4Z^3 - 3Z - \cos 3\alpha = 0 \tag{24}$$

are the real numbers

$$\cos\alpha \quad \cos(\alpha + 120^\circ) \quad \cos(\alpha + 240^\circ) \tag{25}$$

If ϕ is any angle then $\cos 3\phi = \cos(\phi + 2\phi) = \cos\phi\cos 2\phi - \sin\phi\sin 2\phi = \cos\phi(2\cos^2\phi - 1) - 2\sin^2\phi\cos\phi = 2\cos^3\phi - \cos\phi - 2\cos\phi(1 - \cos^2\phi) = 4\cos^3\phi - 3\cos\phi$ Therefore $4\cos^3\phi - 3\cos\phi - \cos 3\phi = 0$ if ϕ is any angle Now if α is any angle then three true equations are obtained by replacing ϕ in turn by α $\alpha + 120^\circ$ $\alpha + 240^\circ$ These three equations are

$$\begin{aligned} 4\cos^3\alpha - 3\cos\alpha - \cos 3\alpha &= 0 \\ 4\cos^3(\alpha + 120^\circ) - 3\cos(\alpha + 120^\circ) - \cos 3(\alpha + 120^\circ) &= 0 \\ 4\cos^3(\alpha + 240^\circ) - 3\cos(\alpha + 240^\circ) - \cos 3(\alpha + 240^\circ) &= 0 \end{aligned} \tag{26}$$

But $\cos 3(\alpha + 120^\circ) = \cos(3\alpha + 360^\circ) = \cos 3\alpha$ and $\cos 3(\alpha + 240^\circ) = \cos(3\alpha + 720^\circ) = \cos 3\alpha$ Hence equations (26) become

$$\begin{aligned} 4(\cos\alpha)^3 - 3\cos\alpha - \cos 3\alpha &= 0 \\ 4[\cos(\alpha + 120^\circ)]^3 - 3\cos(\alpha + 120^\circ) - \cos 3\alpha &= 0 \\ 4[\cos(\alpha + 240^\circ)]^3 - 3\cos(\alpha + 240^\circ) - \cos 3\alpha &= 0 \end{aligned} \tag{27}$$

It is to be noted especially that the third term is the same in each of these equations Also equations (27) state the fact that the numbers (25) are the roots of (24)

It will now be proved that there is an angle α such that

$$(28)\qquad -\frac{\cos 3\alpha}{4} = \frac{D}{(\sqrt{-4C/3})^3}.$$

This can be written in the form

$$(29)\qquad \cos 3\alpha = \frac{-4D}{(\sqrt{-4C/3})^3}.$$

Therefore it is sufficient to show that

$$(30)\qquad -1 < \frac{4D}{(\sqrt{-4C/3})^3} < 1.$$

This continued inequality is true if and only if

$$(31)\qquad -1 < \frac{D}{2(\sqrt{-C/3})^3} < 1.$$

Also $(\sqrt{-C/3})^3 = \sqrt{-C^3/27}$. Therefore (31) is true if and only if $|D| < 2\sqrt{-C^3/27}$, and hence if and only if $D^2 < 4(-C^3/27)$. By (19) this last inequality is true.

It has been proved that there is an angle 3α such that (28) is true. The angle α is obtained by division. If this value of α is used in (25), the resulting real numbers are the roots of (23). If this value of α is used in

$$(32)\qquad \begin{gathered} \sqrt{\frac{-4C}{3}}\cos\alpha, \\ \sqrt{\frac{-4C}{3}}\cos(\alpha + 120°), \\ \sqrt{\frac{-4C}{3}}\cos(\alpha + 240°), \end{gathered}$$

the roots of (18) are obtained.

This completes the proof of the following theorem.

THEOREM 2. *If $y^3 + Cy + D = 0$ is an equation with real coefficients such that $D^2 + 4(C^3/27) < 0$, then $C < 0$. If $D = 0$, its roots are the real numbers $0, +\sqrt{-C}, -\sqrt{-C}$. If $D \neq 0$, then $\sqrt{-4C/3}$ is a positive number and there is an angle α such that (29)*

is true. Then the roots of $y^3 + Cy + D = 0$ *are the real numbers* (32).

In applying theorem 2 to obtain numerical results logarithms should be used. By (29) the sign of $\cos 3\alpha$ is opposite to the sign of D. If $D < 0$, an acute angle 3α is found from

(33) $\log \cos 3\alpha = \log(-D) - 1.5 \log(-C) + 1.5 \log 3 - \log 2$

If $D > 0$, then there is an obtuse angle 3α and there is an acute angle ζ such that

$$(34) \qquad \begin{aligned} 3\alpha &= 180^\circ - \zeta, \\ \log \cos \zeta &= \log D - 1.5 \log(-C) + 1.5 \log 3 - \log 2 \end{aligned}$$

After 3α is found from (33) or (34), then α is obtained by division. Now α is acute since 3α is between 0° and 180°. Therefore $y_1 > 0$ and $\log y_1 = \log \cos \alpha + [\log(-C) + \log 4 - \log 3]/2$. Also $\alpha + 120^\circ$ terminates in the second quadrant and there is an acute angle β such that $\cos(\alpha + 120^\circ) = -\cos \beta$. Therefore $y_2 < 0$ and $\log(-y_2) = \log \cos \beta + [\log(-C) + \log 4 - \log 3]/2$. Again $\alpha + 240^\circ$ terminates either in the third quadrant or in the fourth quadrant. If it terminates in the third quadrant, there is an acute angle γ such that $\cos(\alpha + 240^\circ) = -\cos \gamma$. Then $y_3 < 0$ and $\log(-y_3) = \log \cos \gamma + [\log(-C) + \log 4 - \log 3]/2$. If $\alpha + 240^\circ$ terminates in the fourth quadrant, there is an acute angle δ such that $\cos(\alpha + 240^\circ) = \cos \delta$, $y_3 > 0$ and $\log y_3 = \log \cos \delta + [\log(-C) + \log 4 - \log 3]/2$.

Horner's method, which is discussed in chapter 4, can also be used to compute the roots of equations to which theorem 2 is applicable. Other facts about these equations are discussed in the references at the end of this book.

PROBLEMS

Show that the reduced cubic equation of each of the following equations has $R < 0$. Then solve the equation by the method of theorem 2.

1. $x^3 + 6x^2 + 9x + 1 = 0$	2. $x^3 - 3x^2 - 3x + 3 = 0$
3. $x^3 + 3x^2 - 2x - 5 = 0$	4. $x^3 + 6x^2 + 10x + 3 = 0$
5. $x^3 - 3x^2 - x + 4 = 0$	6. $x^3 + 12x^2 + 43x + 46 = 0$
7. $3x^3 + 9x^2 + 3x - 2 = 0$	8. $2x^3 + 6x^2 - 2x - 5 = 0$
9. $4x^3 - 24x^2 + 44x - 23 = 0$	10. $x^3 - 6x^2 + 6x + 8 = 0$
11. $2x^3 - 6x^2 + x + 2 = 0$	12. $3x^3 - 18x^2 + 31x - 13 = 0$
13. $2x^3 + 12x^2 + 20x + 9 = 0$	14. $2x^3 + 12x^2 + 18x + 5 = 0$
15. $120x^3 + 180x^2 - 190x - 29 = 0$	16. $4x^3 + 36x^2 + 102x + 89 = 0$

4. Discriminant of the cubic equation. By theorem 2 the roots of the reduced cubic equation are all real if $R < 0$. By (3) x is real if and only if y is real. It will now be proved that there are other conditions under which the roots of a cubic are all real. In fact, there is an expression involving the coefficients of the cubic from which the character of the roots of the cubic can be determined without finding the roots. It is to be noted that there is an analogous fact for the quadratic equation $ax^2 + bx + c = 0$. The discriminant of this equation is $b^2 - 4ac$. It is known that the roots are real if and only if the discriminant is greater than or equal to zero.

The discriminant of the reduced cubic equation (6) is designated by Δ. By definition

$$\Delta = (y_1 - y_2)^2(y_1 - y_3)^2(y_2 - y_3)^2.$$

An expression for Δ in terms of the coefficients C and D of (6) will now be obtained. Then later it will be proved that the roots of (6) are all real if and only if $\Delta \geqq 0$.

It will now be proved that

$$\Delta = -4C^3 - 27D^2. \tag{35}$$

If $C = 0$, then (6) becomes $y^3 = -D$, and its roots are $y_1 = \sqrt[3]{-D}$, $y_2 = \omega y_1$, $y_3 = \omega^2 y_1$. Hence $y_1 - y_2 = \sqrt[3]{-D}(1 - \omega)$, $y_1 - y_3 = \sqrt[3]{-D}(1 - \omega^2)$, $y_2 - y_3 = \sqrt[3]{-D}(\omega - \omega^2)$. Hence, by the definition,

$$\Delta = [(\sqrt[3]{-D})^3(1 - \omega)(1 - \omega^2)(\omega - \omega^2)]^2.$$

Also $1 + \omega + \omega^2 = 0$ by the definition of ω, and $\omega^3 = 1$. Therefore $(1 - \omega)(1 - \omega^2) = 3$. Again, $\omega - \omega^2 = \sqrt{3}i$. Finally $(\sqrt[3]{-D}\,)^3 = -D$. Hence $\Delta = (-D\cdot 3\cdot\sqrt{3}i)^2 = D^2\cdot 27i^2 = -27D^2$. Therefore (35) is true if $C = 0$.

If $C \neq 0$, then, by (17), $y_1 - y_2 = (z_1 + z_4) - (\omega z_1 + \omega^2 z_4) = (1 - \omega)(z_1 - \omega^2 z_4)$. Also $y_1 - y_3 = (z_1 + z_4) - (\omega^2 z_1 + \omega z_4) = (1 - \omega^2)(z_1 - \omega z_4)$. Finally $y_2 - y_3 = (\omega z_1 + \omega^2 z_4) - (\omega^2 z_1 + \omega z_4) = (\omega - \omega^2)(z_1 - z_4)$. Hence, by the definition,

$$\Delta = [(1 - \omega)(1 - \omega^2)(\omega - \omega^2)(z_1 - z_4)(z_1 - \omega z_4)(z_1 - \omega^2 z_4)]^2.$$

Now it was proved in the preceding case that $(1 - \omega)(1 - \omega^2) = 3$ and that $\omega - \omega^2 = \sqrt{3}i$. Also it is verified directly that $(z_1 - z_4)(z_1 - \omega z_4)(z_1 - \omega^2 z_4) = z_1^3 - z_4^3$. Hence, by (12) and

(13) this product is $\sqrt{R}$ Therefore $\Delta = (3\sqrt{3}i\sqrt{R})^2 = -27R$ Then (35) follows by (11) This completes the proof of the following theorem

THEOREM 3 *The discriminant Δ of the reduced cubic equation $y^3 + Cy + D = 0$ is by definition the function $(y_1 - y_2)^2(y_1 - y_3)^2(y_2 - y_3)^2$ of its roots y_1 y_2 y_3 The value of Δ in terms of the coefficients of the equation is $-4C^3 - 27D^2$ Also $\Delta = -27R$*

The value of the function $(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2$ of the roots of the general cubic equation (1) will now be determined in terms of the coefficients a b c d of that equation By (3) $x_1 - x_2 = y_1 - y_2$ $x_1 - x_3 = y_1 - y_3$ $x_2 - x_3 = y_2 - y_3$ Also by (5) and (35)

$$(36)\qquad a^4\Delta = -4ac^3 + b^2c^2 - 4b^3d + 18abcd - 27a^2d^2$$

Therefore by the definition of Δ

$$(37)\qquad a^4(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2 = -4ac^3 + b^2c^2 - 4b^3d + 18abcd - 27a^2d^2$$

The left-hand side of (37) is by definition *the discriminant of the general cubic equation (1)* Hence the following result has been proved

THEOREM 4 *The discriminant of the general cubic equation $ax^3 + bx^2 + cx + d = 0$ is by definition the function $a^4(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2$ of its roots x_1 x_2 x_3 The value of this discriminant is $a^4\Delta$ in which Δ is the discriminant of its reduced cubic The value of the discriminant of the equation $ax^3 + bx^2 + cx + d = 0$ in terms of the coefficients of this equation is $-4ac^3 + b^2c^2 - 4b^3d + 18abcd - 27a^2d^2$*

It is especially to be noted that the function $a^4(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2$ is used as the definition of the discriminant of the general cubic instead of the function $(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2$ In section 3 of chapter 9 there is an explanation of this fact

LEMMA 1 *A cubic equation with real coefficients has three real roots or it has one real root and two complex roots which are not real These non-real roots are conjugate complex numbers*

PROOF. Since x is real if and only if y is real, it is sufficient to prove the lemma for (6). It will be proved that, if A and B are real numbers such that $B \neq 0$ and $A + Bi$ is a root of (6), then $A - Bi$ and $-2A$ are the other two roots of (6). By substituting $A + Bi$ in (6) and performing the indicated operations, the equation $A^3 - 3AB^2 + CA + D + (3A^2B - B^3 + CB)i = 0$ is obtained. By the definition of equality of complex numbers, this is true if and only if $A^3 - 3AB^2 + CA + D = 0$ and $3A^2B - B^3 + CB = 0$. These same conditions are obtained if $A - Bi$ is substituted in (6). Again, since $B \neq 0$, the second condition implies $B^2 = 3A^2 + C$. If this is used in the first condition, the equation $-8A^3 - 2CA + D = 0$ is obtained. Therefore $-2A$ is the third root of (6).

THEOREM 5. *The roots of the real reduced cubic equation* (6) *are all real if and only if* $\Delta \geqq 0$.

PROOF. By the definition, $\Delta \geqq 0$ if each of y_1, y_2, y_3 is a real number. This is the meaning of that part of theorem 5 which states that the roots are all real only if $\Delta \geqq 0$.

It will now be proved that, if $\Delta \geqq 0$, then the roots are all real. This will be done by showing that the second of the two possibilities in lemma 1 contradicts the hypothesis $\Delta \geqq 0$. If y_1 is the real root of (6) and y_2 is given the notation $s + ti$, in which s and t are real numbers and $t \neq 0$, then $y_3 = s - ti$. Therefore $y_1 - y_2 = (y_1 - s) - ti$, and $y_1 - y_3 = (y_1 - s) + ti$. Also $y_2 - y_3 = 2ti$. Hence $\Delta = \{[(y_1 - s) - ti][(y_1 - s) + ti]\cdot 2ti\}^2 = [(y_1 - s)^2 + t^2]^2(-4t^2)$. Since t, y_1, s are all real and $t \neq 0$, it is true that $(y_1 - s)^2 \geqq 0$ and $[(y_1 - s)^2 + t^2]^2 \geqq t^4 > 0$. Therefore $\Delta < 0$. This contradicts the hypothesis that $\Delta \geqq 0$.

THEOREM 6. *The roots of a real cubic equation are all real and unequal if and only if its discriminant is positive. At least two of the roots are equal if and only if its discriminant is zero; then all the roots are real. One of the roots is a real number and the other two roots are conjugate complex numbers which are not real if and only if the discriminant is negative.*

PROOF. By (3) and the relation between the discriminant of (1) and the discriminant of (6) which was proved in theorem 4, it is sufficient to prove theorem 6 for (6). By (35) Δ is a real number. Therefore $\Delta > 0$, $\Delta = 0$, or $\Delta < 0$. By the definition,

$\Delta = 0$ if and only if at least two of the roots are equal. Then the roots are all real by theorem 5. Also, the roots are all real and distinct if and only if $\Delta > 0$, by theorem 5. Again, if one of the roots is a real number and the other two roots are conjugate complex numbers which are not real, then $\Delta < 0$ by the proof of theorem 5. This is the meaning of that part of the last sentence of theorem 6 which states that the roots are of this nature only if $\Delta < 0$. Finally it will be proved that, if $\Delta < 0$, then the roots are of this nature. This is true because, by the first sentence in the proof of theorem 5, the first possibility in lemma 1 contradicts the hypotheses that $\Delta < 0$.

PROBLEMS

Compute the discriminant for each of the following equations and characterize its roots.

1. $x^3 - 5x^2 + 8x - 4 = 0$
2. $4x^3 + 4x^2 - 15x - 18 = 0$
3. $x^3 + 2x^2 - 5x - 6 = 0$
4. $x^3 + 4x^2 + x - 6 = 0$
5. $3x^3 + 8x^2 + 7x + 2 = 0$
6. $x^3 + 2x^2 + x - 4 = 0$
7. $x^3 + x^2 - 3x + 9 = 0$
8. $2x^3 - 11x^2 + 12x + 9 = 0$
9. $y^3 + 2y^2 + 12y - 40 = 0$
10. $y^3 + 6y^2 + 12y + 8 = 0$
11. $y^3 + 2y^2 + 8y + 7 = 0$
12. $y^3 + 3y^2 + 24y - 28 = 0$
13. $y^3 + 5y^2 + 20y + 64 = 0$
14. $y^3 + 7y^2 + 16y + 12 = 0$
15. $y^3 - y^2 - 16y - 20 = 0$
16. $y^3 + 33y^2 - 8y - 300 = 0$

5 Algebraic solution of the quartic equation. The roots of an equation of the fourth degree are obtained by solving auxiliary cubic and quadratic equations. A notation for the general quartic equation is

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0, \quad A \neq 0 \tag{38}$$

A real quartic equation is an equation of the form (38) whose coefficients are real numbers. A number r is a root of (38) if and only if r is a root of the equation

$$x^4 + \frac{B}{A}x^3 + \frac{C}{A}x^2 + \frac{D}{A}x + \frac{E}{A} = 0 \tag{39}$$

Hence the general quartic equation may be given the notation

$$x^4 + bx^3 + cx^2 + dx + e = 0 \tag{40}$$

Now (40) is equivalent to the equation

$$x^4 + bx^3 + \left(\frac{b}{2}\right)^2 x^2 = \left(\frac{b}{2}\right)^2 x^2 - cx^2 - dx - e,$$

and hence to

$$\left(x^2 + \frac{b}{2}x\right)^2 = \left(\frac{b^2}{4} - c\right)x^2 - dx - e. \tag{41}$$

It may be that the expression on the right of (41) is a perfect square of a linear function of x. This is true, for example, if the quartic (40) is the equation $x^4 + 4x^3 + 3x^2 - 6x - 9 = 0$. Then the equation (41) is $(x^2 + 2x)^2 = x^2 + 6x + 9$. This equation is equivalent to $(x^2 + 2x)^2 = (x + 3)^2$. Hence the roots of the original numerical quartic are all the roots of $x^2 + 2x = x + 3$ and all the roots of $x^2 + 2x = -(x + 3)$.

If the quadratic function on the right of (41) is the square of a linear function, that is, if there are constants p and q such that

$$\left(\frac{b^2}{4} - c\right)x^2 - dx - e \equiv (px + q)^2, \tag{42}$$

then the roots of (40) are the roots of the two quadratic equations

$$x^2 + \frac{b}{2}x = px + q, \tag{43}$$

$$x^2 + \frac{b}{2}x = -(px + q). \tag{44}$$

Now a quadratic function is the square of a linear function if and only if the roots of the corresponding quadratic equation are equal numbers, and hence if and only if the discriminant of the quadratic function is zero. Also, the discriminant of the quadratic function which forms the left-hand side of (42) is $(-d)^2 - 4[(b^2/4) - c](-e)$. Therefore there are constants p and q such that (42) is true if and only if

$$d^2 + e(b^2 - 4c) = 0. \tag{45}$$

If (45) is true of the coefficients of (40), then the roots of (40) are the roots of the two quadratic equations (43) and (44).

But, if (45) is not true of the coefficients of (40), then the right-hand side of (41) is not the square of a linear function of x, and the preceding method of solving (41) is not applicable

One way of solving (41), if (45) is not true, will now be explained *If t is any number*, then the roots of (41) are the same as the roots of the equation which is obtained by adding $[x^2 + (b/2)x]t + (t^2/4)$ to both sides of (41) Therefore (41) is equivalent to

$$(46)\quad \left(x^2 + \frac{b}{2}x\right)^2 + \left(x^2 + \frac{b}{2}x\right)t + \frac{t^2}{4}$$
$$= \left(\frac{b^2}{4} - c + t\right)x^2 + \left(\frac{b}{2}t - d\right)x + \left(\frac{t^2}{4} - e\right)$$

The left-hand side of (46) is the perfect square $[x^2 + (b/2)x + (t/2)]^2$ Also, by the fact which follows (44) the quadratic function on the right of (46) is the square of a linear function of x if and only if the discriminant of this quadratic function is zero Therefore, *if t is a particular number* such that

$$(47)\qquad \left(\frac{b}{2}t - d\right)^2 - 4\left(\frac{b^2}{4} - c + t\right)\left(\frac{t^2}{4} - e\right) = 0,$$

then the particular equation (46) in which t has this value will have its right-hand side actually the square of a linear function of x That is, if t is a value of y which satisfies the equation

$$(48)\qquad \left(\frac{b}{2}y - d\right)^2 - 4\left(\frac{b^2}{4} - c + y\right)\left(\frac{y^2}{4} - e\right) = 0,$$

then there are numbers P and Q depending on t such that the particular equation (46) in which t has this value becomes

$$(49)\qquad \left(x^2 + \frac{b}{2}x + \frac{t}{2}\right)^2 = (Px + Q)^2$$

Then the roots of (41) are the roots of the two equations

$$(50)\qquad x^2 + \frac{b}{2}x + \frac{t}{2} = Px + Q,$$

$$(51)\qquad x^2 + \frac{b}{2}x + \frac{t}{2} = -(Px + Q)$$

If the operations in (48) are performed and terms involving the same powers of y are combined, the resulting equation is

$$(52) \qquad y^3 - cy^2 + (bd - 4e)y - d^2 - b^2e + 4ce = 0.$$

This equation is called *the resolvent cubic equation of the quartic* (40).

Logically there are not two cases, depending on whether $d^2 + e(b^2 - 4c)$ is or is not zero, because, if $t = 0$ in (46), then (41) is obtained. In practice the expression $d^2 + e(b^2 - 4c)$ is first computed. If $d^2 + e(b^2 - 4c)$ is zero, then (42) is written down at once, with no reference to (52). But if $d^2 + e(b^2 - 4c)$ is not zero, then one root of (52) is found. Then equation (49) is written down. In each case the four roots of the quartic equation are the roots of two quadratic equations. The following theorem has been proved.

THEOREM 7. *The roots of the quartic equation $x^4 + bx^3 + cx^2 + dx + e = 0$ are found from a root t of the resolvent cubic equation $y^3 - cy^2 + (bd - 4e)y - d^2 - b^2e + 4ce = 0$. There are numbers P and Q such that the quadratic function $[(b^2/4) - c + t]x^2 + [(b/2)t - d]x + [(t^2/4) - e]$ is $(Px + Q)^2$. Then the roots of the quartic equation are the roots of the two quadratic equations (50) and (51).*

In the following problems each equation has the form (40) with integral coefficients. Therefore the resolvent cubic equation has integral coefficients. The following illustration shows how to determine any integral root which the cubic may have. The resolvent cubic of $x^4 - 3x^2 + 1 = 0$ is $y^3 + 3y^2 - 4y - 12 = 0$. If k is an integer which satisfies this cubic equation, then $k^3 + 3k^2 - 4k = 12$. Therefore $k(k^2 + 3k - 4) = 12$. This shows that k is a factor of 12. If one of the integers in the list ± 1, ± 2, ± 3, ± 4, ± 6, ± 12 satisfies the cubic, it is used as t. If each integer in the list does not satisfy the cubic, then the cubic has no integral root, and a non-integral root is used as t. Similarly, an integral root of (52) is a factor of the constant term of (52). If one of the factors of the constant term of (52) satisfies (52), it is used as t. If each of these factors does not satisfy (52), then (52) has no integral root and a non-integral root of (52) is used as t.

PROBLEMS

Solve the following quartic equations by the method of theorem 7

1 $x^4 - 2x^2 + 8x - 3 = 0$
2 $x^4 - 6x^2 + 8x - 3 = 0$
3 $x^4 - 2x^3 + 3x - 2 = 0$
4 $x^4 - 3x^2 - 10x - 6 = 0$
5 $x^4 - 5x^2 - 6x - 5 = 0$
6 $x^4 - 7x^2 + 10x - 4 = 0$
7 $x^4 + x^2 + 6x + 4 = 0$
8 $x^4 + 5x - 6 = 0$
9 $x^4 + 3x - 2 = 0$
10 $x^4 - 5x^2 + 6x + 3 = 0$
11 $x^4 - 8x^2 - 15x - 6 = 0$
12 $x^4 - x^2 + 2x + 2 = 0$
13 $x^4 - 10x^2 + 9x - 2 = 0$
14 $x^4 - 12x^2 - 3x + 2 = 0$
15 $x^4 - 33x^2 - 6x + 2 = 0$
16 $x^4 - 14x^2 + 3x + 6 = 0$
17 $4x^4 + 4x^3 + 15x^2 + 8 = 0$
18 $12x^4 + 24x^3 + 32x^2 + 12x + 3 = 0$
19 $3x^4 - 3x^3 + 4x^2 - 3x + 3 = 0$
20 $x^4 + 3x^3 + 4x^2 + 1 = 0$

6 Discriminant of the quartic equation If x_1 and x_2 are the roots of (50) and x_3 and x_4 the roots of (51) then x_1 x_2 x_3 x_4 are the roots of (40) Now it is proved in elementary algebra that if x_1 and x_2 are the roots of (50) and hence of

$$x^2 + \left(\frac{b}{2} - P\right)x + \left(\frac{t}{2} - Q\right) = 0 \tag{53}$$

then

$$x^2 + \left(\frac{b}{2} - P\right)x + \left(\frac{t}{2} - Q\right) = (x - x_1)(x - x_2) \tag{54}$$

Similarly if x_3 and x_4 are the roots of (51) and hence of

$$x^2 + \left(\frac{b}{2} + P\right)x + \left(\frac{t}{2} + Q\right) = 0 \tag{55}$$

then

$$x^2 + \left(\frac{b}{2} + P\right)x + \left(\frac{t}{2} + Q\right) = (x - x_3)(x - x_4) \tag{56}$$

Equation (49) can be written in the form $[x^2 + (b/2)x + (t/2)]^2 - (Px + Q)^2 = 0$ Similarly (46) can be written in the form

$f(x) = 0$. Therefore, for the particular t used in (49), $f(x)$ is identically equal to $[x^2 + (b/2)x + (t/2)]^2 - (Px + Q)^2$. Also, by the way in which (46) was obtained from (10),

$$x^4 + bx^3 + cx^2 + dx + e = f(x). \tag{57}$$

Therefore

$$x^4 + bx^3 + cx^2 + dx + e = \left(x^2 + \frac{b}{2}x + \frac{t}{2}\right)^2 - (Px + Q)^2. \tag{58}$$

The function on the right-hand side of (58) is the product of the function on the left-hand side of (54) and that on the left-hand side of (56). Therefore

$$\begin{aligned} x^4 + bx^3 + cx^2 + dx + e \\ = (x - x_1)(x - x_2)(x - x_3)(x - x_4). \end{aligned} \tag{59}$$

The expanded form of the product on the right-hand side of (59) is $x^4 - (x_1 + x_2 + x_3 + x_4)x^3 + (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)x^2 - (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)x + x_1x_2x_3x_4$. Hence, by equating the coefficients of like powers of x, the following relations are obtained:

$$\begin{aligned} -b &= x_1 + x_2 + x_3 + x_4, \\ c &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4, \\ -d &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4, \\ e &= x_1x_2x_3x_4. \end{aligned} \tag{60}$$

In the following discussion the particular functions $x_1x_2 + x_3x_4$, $x_1x_3 + x_2x_4$, $x_1x_4 + x_2x_3$ of the roots x_1, x_2, x_3, x_4 of the quartic occur frequently. They will therefore be designated by z_1, z_2, z_3. Thus, by definition,

$$\begin{aligned} z_1 &= x_1x_2 + x_3x_4, \\ z_2 &= x_1x_3 + x_2x_4, \\ z_3 &= x_1x_4 + x_2x_3. \end{aligned} \tag{61}$$

Now, if these three equations are added and the second equation in (60) is used, it is found that

$$z_1 + z_2 + z_3 = c. \tag{62}$$

Another important relation between z_1, z_2, z_3 will now be found. Thus $z_1z_2 + z_1z_3 + z_2z_3 = (x_1x_2 + x_3x_4)(x_1x_3 + x_2x_4) + (x_1x_2 +$

LEMMA 3 *The four roots of a real quartic equation have one of the following properties (i) all the roots are real numbers, (ii) two roots are real numbers, and two roots are conjugate complex numbers which are not real, (iii) the four roots are two pairs of conjugate complex numbers which are not real*

PROOF First it will be proved that, if s and t are real numbers such that $t \neq 0$ and $s + ti$ is a root of (40), then $s - ti$ is a root of (40) Substitution of $s + ti$ in (40) yields an equation of the form $c_1 + c_2 i = 0$ in which c_1 and c_2 are real constants depending on b, c, d, e, s t Substitution of $s - ti$ in (40) yields the equation $c_1 - c_2 i = 0$ By the definition of equality of complex numbers each of these equations is true if and only if $c_1 = 0$ and $c_2 = 0$ Again, using $t \neq 0$ and the expressions for c_1 and c_2 it is verified that $(x^2 - 2sx + s^2 + t^2)[x^2 + (b + 2s)x + c + 3s^2 - t^2 + 2sb] = x^4 + bx^3 + cx^2 + dx + e$ The roots of the equation formed by equating the first factor to zero are $s + ti$ and $s - ti$ The roots of the equation formed by equating the second factor to zero are real numbers or they are conjugate complex numbers which are not real

THEOREM 9 *There are at least two equal roots of a real quartic equation if and only if its discriminant is zero No two of the roots are equal and two roots are real while two roots are conjugate complex numbers which are not real if and only if the discriminant of the equation is negative The four roots are real and unequal or the four roots are unequal and form two pairs of conjugate complex numbers which are not real, if and only if the discriminant of the equation is positive*

are verified in the same way. Hence, by (65), $\delta = (y_1 - y_2)^2(y_1 - y_3)^2(y_2 - y_3)^2$. Again, the discriminant of the resolvent cubic (52) is $(y_1 - y_2)^2(y_1 - y_3)^2(y_2 - y_3)^2$, since the leading coefficient of this cubic is 1. Therefore *the discriminant δ of the quartic (40) equals the discriminant of its resolvent cubic (52).*

Comparison of the coefficients of the cubic in theorem 4 and the coefficients of (52) shows that a, b, c, d in theorem 4 are replaced respectively by 1, $-c$, $bd - 4e$, $-d^2 - b^2e + 4ce$ in (52). Therefore the discriminant of (52) is obtained from the expression for the discriminant in theorem 4 by these replacements. Hence

$$(68) \quad \delta = -4(bd - 4e)^3 + c^2(bd - 4e)^2 + 4c^3(-d^2 - b^2e + 4ce) - 18c(bd - 4e)(-d^2 - b^2e + 4ce) - 27(-d^2 - b^2e + 4ce)^2.$$

Another method of obtaining (68) is to find the reduced cubic equation $Y^3 + CY + D = 0$ for the resolvent cubic (52). Thus by (5), with a, b, c, d replaced respectively by 1, $-c$, $bd - 4e$, $-d^2 - b^2e + 4ce$,

$$(69) \quad C = bd - 4e - \frac{c^2}{3}, \qquad D = -d^2 - b^2e + \frac{8ce}{3} + \frac{bcd}{3} - \frac{2c^3}{27},$$

in the reduced cubic of (52). By theorem 3, the discriminant of this reduced cubic equation is $-4C^3 - 27D^2$. Since the coefficient of y^3 in (52) is 1, the discriminant of the resolvent cubic (52) equals the discriminant of its reduced cubic. Therefore

$$(70) \quad \delta = -4C^3 - 27D^2,$$

in which the values of C and D are given by (69). This completes the proof of the following theorem.

THEOREM 8. *Let x_1, x_2, x_3, x_4 be the roots of the quartic equation $x^4 + bx^3 + cx^2 + dx + e = 0$. Then its discriminant δ is, by definition, the function*

$$(x_1 - x_2)^2(x_1 - x_3)^2(x_1 - x_4)^2(x_2 - x_3)^2(x_2 - x$$

Also δ is evaluated in terms of the coefficients of t

and (69). The discriminant δ of the quartic (

criminant of its resolvent cubic equation (52).

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by showing that
ld a contradiction,
nesis $\delta < 0$ yield a

LEMMA 3 *The four roots of a real quartic equation have one of the following properties (i) all the roots are real numbers, (ii) two roots are real numbers, and two roots are conjugate complex numbers which are not real, (iii) the four roots are two pairs of conjugate complex numbers which are not real*

PROOF First it will be proved that, if s and t are real numbers such that $t \neq 0$ and $s + ti$ is a root of (40), then $s - ti$ is a root of (40) Substitution of $s + ti$ in (40) yields an equation of the form $c_1 + c_2 i = 0$ in which c_1 and c_2 are real constants depending on b, c, d, e, s, t Substitution of $s - ti$ in (40) yields the equation $c_1 - c_2 i = 0$ By the definition of equality of complex numbers, each of these equations is true if and only if $c_1 = 0$ and $c_2 = 0$ Again using $t \neq 0$ and the expressions for c_1 and c_2, it is verified that $(x^2 - 2sx + s^2 + t^2)[x^2 + (b + 2s)x + c + 3s^2 - t^2 + 2sb] = x^4 + bx^3 + cx^2 + dx + e$ The roots of the equation formed by equating the first factor to zero are $s + ti$ and $s - ti$ The roots of the equation formed by equating the second factor to zero are real numbers, or they are conjugate complex numbers which are not real

THEOREM 9 *There are at least two equal roots of a real quartic equation if and only if its discriminant is zero No two of the roots are equal and two roots are real while two roots are conjugate complex numbers which are not real if and only if the discriminant of the equation is negative The four roots are real and unequal, or the four roots are unequal and form two pairs of conjugate complex numbers which are not real if and only if the discriminant of the equation is positive*

such that $v \neq 0$ and $x_1 = u + vi$. If the notation is chosen so that x_2 is the conjugate of x_1, then $x_2 = u - vi$. Also, there are real numbers s and t such that $t \neq 0$ and $x_3 = s + ti$. Then $x_4 = s - ti$. Now $x_1 - x_2 = 2vi$, and $x_3 - x_4 = 2ti$. Hence $(x_1 - x_2)^2(x_3 - x_4)^2 = 16v^2t^2 > 0$. Again $x_1 - x_3 = (u - s) + (v - t)i$, and $x_2 - x_4 = (u - s) - (v - t)i$. Hence $(x_1 - x_3)(x_2 - x_4) = (u - s)^2 + (v - t)^2$. Since $x_1 \neq x_3$ and $x_2 \neq x_4$, therefore $(u - s)^2 > 0$ or $(v - t)^2 > 0$, and $(x_1 - x_3)^2(x_2 - x_4)^2 > 0$. Similarly it is proved that $(x_1 - x_4)^2(x_2 - x_3)^2 > 0$. Hence by (65) it follows that $\delta > 0$. This completes the proof of the part of the last sentence of theorem 9 which states that the four roots are real and unequal, or the four roots are unequal and form two pairs of conjugate complex numbers which are not real, only if $\delta > 0$. The other part of the last sentence of theorem 9 is the converse of the part which has just been proved. This converse will be proved later.

contradiction Thus by that part of the last sentence of theorem 9 which has already been proved, if no two roots are equal and if (i) is true, then $\delta > 0$ and there is a contradiction of the hypothesis that $\delta < 0$ Similarly the hypothesis (iii) and the hypothesis $\delta < 0$ yield a contradiction It follows that, if $\delta < 0$, then (ii) is true This completes the proof of the other part of the second sentence of theorem 9

The other part of the last sentence of theorem 9 is proved in an analogous manner This completes the proof of theorem 9

The two cases in the last sentence of theorem 9 can be characterized if new functions of the coefficients are used These functions are the Sturm functions for the quartic equation In chapter 4 Sturm's method will be discussed for an equation of arbitrary degree

Discriminants are discussed further in chapter 9 The works listed as references at the end of this book contain additional information about cubic and quartic equations

PROBLEMS

Compute the discriminant for each of the following equations and characterize its roots

1 $x^4 + 5x^3 + 5x^2 - 5x - 6 = 0$
2 $x^4 + 5x^3 - 7x^2 - 29x + 30 = 0$
3 $9x^4 - 10x^3 + 9x - 18 = 0$
4 $x^4 + 5x^3 + x^2 + x - 1 = 0$
5 $12x^4 + 24x^3 + 32x^2 + 12x + 3 = 0$
6 $3x^4 - 3x^3 + 4x^2 - 3x + 3 = 0$
7 $x^4 - 6x^3 + 9x^2 + 4x - 12 = 0$
8 $x^4 + 9x^3 + 27x^2 + 31x + 12 = 0$
9 $x^4 + x^3 + x^2 - 5 = 0$
10 $x^4 + 2x^2 - x - 1 = 0$
11 $x^4 + 4x^3 + 10x^2 + 12x + 9 = 0$
12 $x^4 + 4x^3 + 14x^2 + 20x + 25 = 0$

CHAPTER 3

GENERAL THEOREMS ON ROOTS OF POLYNOMIAL EQUATIONS

1. Integral roots of polynomial equations whose coefficients are integers. Synthetic substitution. The equation

$$(1) \qquad x^4 - 6x^3 - 13x^2 + 2x - 28 = 0$$

will now be used to illustrate an important method in the solution of certain equations. The function of x which constitutes the left-hand side of (1) is called a polynomial in x of degree 4. Let $f(x)$ designate this polynomial. Thus

$$(2) \qquad f(x) \equiv x^4 - 6x^3 - 13x^2 + 2x - 28.$$

It is to be noted especially that each of the coefficients in $f(x)$ is an integer. It will now be explained how to determine whether there is any integer which is a root of (1). Indeed all integral roots of (1) will be determined. There are infinitely many integers. Therefore it is impossible to test each integer by substitution in (1). In fact, no integer should be tested in (1) until some preliminary information is obtained as a guide to the selection of integers to be tested. Furthermore, the test should not be made by direct substitution, because raising an integer to a power is tedious. The test should be made by synthetic substitution, which will be explained later.

If r and s are integers, the statement that *s is a factor of r* means that there is an integer t such that $r = st$. It is also said that *s divides r* and that *s is a divisor of r.*

It will now be proved that, if b is an integer which is a root of (1), then b divides 28. This is the preliminary information to which reference was just made. After this is proved, then the integers ± 1, ± 2, ± 4, ± 7, ± 14, ± 28 could be tested. The integer 3 would not be tested because, if 3 were a root of (1), then there would be an integer q such that $3q = 28$, and hence a contradic-

tion would be obtained The hypothesis that b is a root of (1) means that the number $f(b)$ that is the number

$$b^4 - 6b^3 - 13b^2 + 2b - 28 \tag{3}$$

is indeed zero Now since b is an integer and since each coefficient in (3) is an integer it is true that (3) is an integer Hence

$$b(b^3 - 6b^2 - 13b + 2) = 28 \tag{4}$$

is a true relation between integers Also the number $b^3 - 6b^2 - 13b + 2$ is an integer It will be designated by q Then (4) becomes

$$bq = 28 \tag{5}$$

Therefore b is a factor of 28

Before synthetic substitution is explained theorem 1 will be proved so that later the type of argument which led to (5) may be applied to the general equation instead of merely to the particular equation (1) The integers are the numbers $0 \pm 1 \pm 2 \pm 3$ A polynomial in x of degree n is a function of x which has a very special form It is a sum of one or more terms Each of these terms is the product of a coefficient which does not involve x and does not depend on x and a power of x The exponent of this power of x must be a positive integer or zero The highest power of x has the exponent n and the coefficient of this term is not zero Thus *a polynomial in x of degree n* is an express on of the form

$$a_0x^n + a_1x^{n-1} + \quad + a_{n-1}x + a_n \tag{6}$$

in which n is a positive integer or zero a_0 a_n are independent of x and $a_0 \neq 0$ if $n > 0$ A *real polynomial* is a polynomial whose coefficients are real numbers It is to be noted especially that a constant is a polynomial in x of degree zero If y is independent of x then $yx^2 + 2yx + y^3$ is a polynomial in x of degree 2

THEOREM 1 *If $f(x)$ is a polynomial in x of positive degree n whose coefficients are integers and if b is an integer which is a root of the equation $f(x) = 0$ then b is a factor of the constant term in $f(x)$*

PROOF The statement that b is a root of $f(x) = 0$ means that $f(b) = 0$ that is that $a_0b^n + a_1b^{n-1} + \quad + a_{n-1}b + a_n = 0$

Hence $b(a_0b^{n-1} + a_1b^{n-2} + \cdots + a_{n-1}) = -a_n$. Now, if q is defined by $q = -(a_0b^{n-1} + a_1b^{n-2} + \cdots + a_{n-1})$, then q is an integer such that $bq = a_n$. This completes the proof of theorem 1.

If $f(x)$ is the particular polynomial (2), then

$$f(4) = 1\cdot 4^4 - 6\cdot 4^3 - 13\cdot 4^2 + 2\cdot 4 - 28. \tag{7}$$

The computation of this integer $f(4)$ will now be explained in greater detail than is used in practice, to illustrate the reason why the method is correct. Later this computation will be greatly abbreviated in a simple table. Thus

$$-2 = 1\cdot 4 - 6,$$

$$-21 = -2\cdot 4 - 13 = (1\cdot 4 - 6)4 - 13 = 1\cdot 4^2 - 6\cdot 4 - 13,$$

$$\begin{aligned} -82 = -21\cdot 4 + 2 &= (1\cdot 4^2 - 6\cdot 4 - 13)4 + 2 \\ &= 1\cdot 4^3 - 6\cdot 4^2 - 13\cdot 4 + 2, \end{aligned}$$

$$\begin{aligned} -356 = -82\cdot 4 - 28 &= (1\cdot 4^3 - 6\cdot 4^2 - 13\cdot 4 + 2)4 - 28 \\ &= 1\cdot 4^4 - 6\cdot 4^3 - 13\cdot 4^2 + 2\cdot 4 - 28. \end{aligned}$$

Therefore $f(4) = -356$. It is to be noted especially that each step consists of a multiplication by 4, followed by an addition or subtraction. This process is displayed in the table

$$\begin{array}{rrrrr|l} 1 & -6 & -13 & 2 & -28 & 4 \\ & 4 & -8 & -84 & -328 & \\ \hline 1 & -2 & -21 & -82 & -356 & \end{array}$$

The integer 4 is not a root of (1), since $f(4) \neq 0$.

If the process of computing $f(-2)$ is exhibited similarly in detail, it is found that $f(-2) = -20$. The successive steps yield the table

$$\begin{array}{rrrrr|l} 1 & -6 & -13 & 2 & -28 & -2 \\ & -2 & 16 & -6 & 8 & \\ \hline 1 & -8 & 3 & -4 & -20 & \end{array}$$

The integer -2 is not a root of (1). Similarly it is found that each of $f(1)$, $f(-1)$, $f(2)$, $f(-4)$, $f(7)$, $f(-7)$, $f(14)$, $f(-14)$ is different from zero. Therefore (1) has no integral root.

PROBLEMS

Find all the integral roots of the following equations

1. $x^3 - 2x^2 - 5x + 6 = 0$
2. $x^3 + 3x^2 - 6x - 8 = 0$
3. $x^4 - 7x^3 + 5x^2 + 31x - 30 = 0$
4. $x^4 - 3x^3 - 27x^2 - 13x + 42 = 0$
5. $x^4 + 3x^3 - 2x^2 + 6x - 8 = 0$
6. $x^4 - 5x^3 + 3x^2 + 15x - 18 = 0$
7. $x^4 + 3x^3 - 6x^2 - 14x + 12 = 0$
8. $x^4 - 4x^3 + 5x^2 - 2x - 12 = 0$
9. $x^4 - 2x^3 - 13x^2 + 38x - 24 = 0$
10. $x^4 - 6x^3 + 3x^2 + 26x - 24 = 0$
11. $x^4 + x^3 - 2x^2 + 17x - 5 = 0$
12. $x^4 - x^3 - 4x^2 + 9x - 3 = 0$
13. $x^5 - 9x^4 + 18x^3 + 13x^2 - 19x - 4 = 0$
14. $x^5 - 4x^4 + x^3 + 10x^2 - 20x + 24 = 0$
15. $x^4 - 2x^3 - 39x^2 + 8x + 140 = 0$
16. $x^4 + 8x^3 - 19x^2 - 122x + 240 = 0$
17. $6x^4 - 47x^3 + 63x^2 + 20x - 12 = 0$
18. $9x^4 - 36x^3 - 7x^2 + 30x - 8 = 0$
19. $6x^4 - 7x^3 - 16x^2 + 21x - 6 = 0$
20. $15x^4 + x^3 + 43x^2 + 3x - 6 = 0$

There are several simplifications of this method of finding all the integral roots of a polynomial equation whose coefficients are integers. Before these simplifications are explained it will be proved that the method of synthetic substitution is valid for an arbitrary polynomial (6) in which $n > 0$. The notation $f(x)$ will designate this general polynomial. It is to be noted especially that in this proof the coefficients in (6) are not necessarily integers. Indeed the coefficients may be any complex numbers. Also in this proof the number c is a fixed but arbitrary complex number.

The general rule for computation of $f(c)$ by synthetic substitution will now be explained. The numbers $a_0, a_1, \dots, a_n$ constitute the first row in a table. The numbers in the third row are designated by $k_0, k_1, \dots, k_n$. Under a_1 in the first row k_0c is written in the second row, under a_2 is written $k_1c, \dots,$ under a_n is written $k_{n-1}c$. Also $k_0 = a_0$ and each of $k_1, \dots, k_n$ is the sum of the two numbers standing above it. Thus the table for the computation by synthetic substitution is

$$
\begin{array}{cccccc|l}
a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n & \underline{c} \\
 & k_0c & k_1c & \cdots & k_{n-2}c & k_{n-1}c & \\
\hline
k_0 & k_1 & k_2 & \cdots & k_{n-1} & k_n &
\end{array}
$$

and

$$
(8) \qquad \begin{aligned}
k_0 &= a_0, \\
k_1 &= a_1 + k_0c, \\
k_2 &= a_2 + k_1c, \\
&\ \ \vdots \\
k_n &= a_n + k_{n-1}c.
\end{aligned}
$$

The rule states that the number k_n which is obtained in this manner is indeed $f(c)$.

This rule will now be proved by mathematical induction. First, it will be verified that the rule is correct if $n = 1$. Thus, if $n = 1$, the table has the form

$$
\begin{array}{cc|l}
a_0 & a_1 & \underline{c} \\
 & k_0c & \\
\hline
k_0 & k_1 &
\end{array}
$$

Also, by (8), $k_0 = a_0$ and $k_1 = a_1 + k_0c$. Therefore $k_1 = a_1 + a_0c$. On the other hand, since $n = 1$, it is true by (6) that $f(x) \equiv a_0x + a_1$. Therefore $f(c) = a_0c + a_1$. Therefore $k_1 = f(c)$. Therefore the rule is correct if $n = 1$.

Lemma for the induction. *If n_0 is a value of n such that the method of synthetic substitution is valid for all polynomials of degree n_0, then the method is valid for all polynomials of degree $n_0 + 1$.*

Proof. Let $F(x)$ designate an arbitrary polynomial

$$(9) \qquad A_0x^{n_0+1} + A_1x^{n_0} + \cdots + A_{n_0}x + A_{n_0+1}$$

of degree $n_0 + 1$. Then

$$(10) \quad F(c) = (A_0c^{n_0} + A_1c^{n_0-1} + \cdots + A_{n_0-1}c + A_{n_0})c + A_{n_0+1}.$$

Let $g(x)$ designate the polynomial

$$(11) \qquad A_0x^{n_0} + A_1x^{n_0-1} + \cdots + A_{n_0-1}x + A_{n_0}.$$

Then

$$(12) \qquad g(c) = A_0c^{n_0} + A_1c^{n_0-1} + \cdots + A_{n_0-1}c + A_{n_0}.$$

Hence, by (10),

$$(13) \qquad F(c) = g(c)\cdot c + A_{n_0+1}.$$

The table of synthetic substitution for $F(x)$ is

$$\begin{array}{cccccc|l} A_0 & A_1 & A_2 & A_{n_0-1} & A_{n_0} & A_{n_0+1} & c \\ & k_0c & k_1c & k_{n_0-2}c & k_{n_0-1}c & k_{n_0}c & \\ \hline k_0 & k_1 & k_2 & k_{n_0-1} & k_{n_0} & k_{n_0+1} & \end{array}$$

Also by (11) the table of synthetic substitution for $g(x)$ is precisely this table with its last column deleted By hypothesis the method of synthetic substitution is valid for all polynomials of degree n_0 Also $g(x)$ is a polynomial of degree n_0 Therefore, by the table of synthetic substitution for $g(x)$ it is true that $k_{n_0} = g(c)$ Again by the statement of the rule for synthetic substitution it is true, in the table for $F(x)$, that $k_{n_0+1} = k_{n_0}c + A_{n_0+1}$ Therefore $k_{n_0+1} = g(c)\, c + A_{n_0+1}$ Hence, by (13), it is true that $F(c) = k_{n_0+1}$ This completes the proof of the lemma for the induction

Since it has been verified that the rule for synthetic substitution is valid for polynomials of degree 1 it is known by the lemma that the rule is valid for polynomials of degree 2 Then by the lemma the rule is valid for polynomials of degree 3 Continuation of this process shows that if n is a positive integer, then the rule is valid for polynomials of degree n

It is to be noted especially that there must be $n + 1$ columns in the table if the polynomial has degree n For example, the polynomial $x^5 - 2x^4 + x^2 - 7x + 13$ is written in the form $x^5 - 2x^4 + 0x^3 + x^2 - 7x + 13$ Then the six coefficients in the first line of the table are 1 -2 0 1, -7 13

2 The factor theorem and the remainder theorem Factored form of a polynomial There is an important identity involving the polynomial $f(x)$ which can be written down from the table showing the synthetic substitution for $f(c)$ This identity is the basis for a simplification in the process of finding all integral roots of a polynomial equation with integral coefficients This identity also leads to other theorems of importance in the solution of equations

If $f(x)$ is the particular polynomial (2) and $c = 1$ this identity may be obtained in the following way When the indicated operations are performed it is found that $x^4 - 6x^3 - 13x^2 + 2x - 28 - x^3(x - 1)$ reduces to $-5x^3 - 13x^2 + 2x - 28$ If $f_1(x)$ is defined by

$$(14) \qquad f_1(x) = -5x^3 - 13x^2 + 2x - 28$$

then

(15) $$f(x) \equiv x^3(x-1) + f_1(x).$$

Again, $f_1(x) + 5x^2(x-1)$ reduces to $-18x^2 + 2x - 28$. If $f_2(x)$ is defined by

(16) $$f_2(x) = -18x^2 + 2x - 28,$$

then

(17) $$f_1(x) \equiv -5x^2(x-1) + f_2(x).$$

Therefore by (15)

(18) $$f(x) \equiv x^3(x-1) - 5x^2(x-1) + f_2(x).$$

Again, $f_2(x) + 18x(x-1)$ reduces to $-16x - 28$. If $f_3(x)$ is defined by

(19) $$f_3(x) \equiv -16x - 28,$$

then $f_2(x) \equiv -18x(x-1) + f_3(x)$, and

(20) $$f(x) \equiv (x^3 - 5x^2 - 18x)(x-1) + f_3(x).$$

In the same way

(21) $$f_3(x) \equiv -16(x-1) - 44,$$

and

(22) $$f(x) \equiv (x^3 - 5x^2 - 18x - 16)(x-1) - 44.$$

The polynomial $x^3 - 5x^2 - 18x - 16$ in (22) will be designated by $q(x)$ and the number -44 by r. It has been proved, if $f(x)$ is the polynomial (2) and $c = 1$, that there is a polynomial $q(x)$ and a number r such that

(23) $$f(x) \equiv q(x) \cdot (x - c) + r.$$

This is the important identity which was mentioned at the beginning of this section.

There is a more simple method of obtaining $q(x)$ and r. The table for the computation of $f(1)$ by synthetic substitution is

$$\begin{array}{rrrrr|l} 1 & -6 & -13 & 2 & -28 & \underline{1} \\ & 1 & -5 & -18 & -16 & \\ \hline 1 & -5 & -18 & -16 & -44 & \end{array}$$

The coefficients of $q(x)$ appear in order as the entries in the last line of the table, and the number r is the last entry in that line.

THEOREM 2 *If n is a positive integer if $f(x)$ is a polynomial in x of degree n and if c is a constant then there is a polynomial $q(x)$ of degree $n - 1$ and a constant r such that $f(x) = (x - c)q(x) + r$ Also $r = f(c)$ Therefore $f(x) \equiv (x - c)q(x) + f(c)$*

PROOF The theorem will be proved by mathematical induction If $n = 1$ then $f(x)$ is $a_0x + a_1$ Therefore $q(x) \equiv a_0$ and $r \equiv a_0c + a_1$ Also $r = f(c)$

LEMMA FOR THE INDUCTION *If n_0 is a value of n such that the theorem is true for polynomials whose degrees are at most n_0, then $n_0 + 1$ is a value of n for which the theorem is true*

PROOF OF THE LEMMA If $F(x)$ is the polynomial (9) a polynomial $Q(x)$ of degree n_0 and a constant R will be found for which

(24) $$F(x) = (x - c)Q(x) + R$$

If $F_1(x)$ means the polynomial $A_0cx^{n_0} + A_1x^{n_0} + \quad + A_{n_0}x + A_{n_0+1}$ then $F(x) = (x - c)x^{n_0}A_0 + F_1(x)$ If $F_1(x)$ is constant then $Q(x) = x^{n_0}A_0$ $R = F_1(x)$ Also then $F(c) = F_1(c)$ and

(25) $$R = F(c)$$

If $F_1(x)$ is not a constant then the theorem is true for $F_1(x)$ by the hypothesis of the lemma Therefore there is a polynomial $Q_1(x)$ and a constant R_1 such that $F_1(x) \equiv (x - c)Q_1(x) + R_1$ and $F_1(c) = R_1$ Then $F(x) \equiv (x - c)[x^{n_0}A_0 + Q_1(x)] + R_1$ $Q(x) = x^{n_0}A_0 + Q_1(x)$ $R = R_1$ Also $F(c) = R_1$ Therefore (25) is true

Verification if $n = 1$ and proof of the lemma for the induction complete the proof of theorem 2

Each identity in theorem 2 is called *the division algorithm for $f(x)$ and c* although the only operations in the identities are multiplication addition and subtraction If x in the identity is replaced by any constant an equation between numbers is obtained The polynomial $q(x)$ is called the quotient and r is called the remainder Theorem 2 is called *the remainder theorem*

The statement that *a polynomial $s(x)$ is a factor of a polynomial $t(x)$* means that there is a polynomial $q(x)$ such that $t(x) = q(x)s(x)$ It is also said that $s(x)$ is a divisor of $t(x)$ and that $s(x)$ divides $t(x)$ Theorem 3 about factors is a corollary of theorem 2 and is called *the factor theorem*

THEOREM 3. *If n is a positive integer, if $f(x)$ is a polynomial in x of degree n, and if c is a root of the equation $f(x) = 0$, then there is a polynomial $q(x)$ of degree $n - 1$ such that $f(x) \equiv (x - c)q(x)$. Thus $x - c$ is a factor of $f(x)$.*

PROOF. If c is a root of $f(x) = 0$, then $f(c) = 0$, and the division algorithm becomes $f(x) \equiv (x - c)q(x)$. This is the result in theorem 3.

It will now be proved for the general polynomial (6) of positive degree that the coefficients in $q(x)$ in the division algorithm are indeed the entries in the third line of the table for the computation of $f(c)$ by synthetic substitution. Let $q(x)$ be given the notation

$$q(x) \equiv b_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-2}x + b_{n-1}. \tag{26}$$

Then

$$(b_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-2}x + b_{n-1})(x - c) + r \tag{27}$$

becomes precisely (6), after the indicated operations are performed. On the other hand, after these operations are performed, (27) becomes

$$\begin{aligned} b_0x^n + (-cb_0 + b_1)x^{n-1} + (-cb_1 + b_2)x^{n-2} \\ + \cdots + (-cb_{n-2} + b_{n-1})x + (-cb_{n-1} + r). \end{aligned} \tag{28}$$

Therefore, by equating coefficients in (6) and (28), the numbers $b_0, \cdots, b_{n-1}, r$ satisfy the equations $b_0 = a_0$, $-cb_0 + b_1 = a_1$, $-cb_1 + b_2 = a_2$, $\cdots$, $-cb_{n-2} + b_{n-1} = a_{n-1}$, $-cb_{n-1} + r = a_n$. These equations are equivalent to

$$\begin{aligned} b_0 &= a_0 \\ b_1 &= a_1 + b_0c \\ b_2 &= a_2 + b_1c \\ &\cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \\ b_{n-1} &= a_{n-1} + b_{n-2}c \\ r &= a_n + b_{n-1}c. \end{aligned} \tag{29}$$

Hence, by (8), $b_0 = k_0$, $b_1 = k_1$, $\cdots$, $b_{n-1} = k_{n-1}$. This completes the proof of the following theorem.

THEOREM 4 *If n is a positive integer, if $f(x)$ is a polynomial in x of degree n, if c is a constant, if $q(x)$ is a polynomial and r a constant such that $f(x) \equiv (x - c)q(x) + r$, and if $q(x)$ has the notation (26), then the numbers $b_0, b_1, \ldots, b_{n-1}, r$ are precisely the n entries in the last line of the table for the computation of $f(c)$ by synthetic substitution*

PROBLEMS

In each of the following problems for c and $f(x)$ as stated find $q(x)$ and r, and write the division transformation identity

1 $2, x^3 - 3x^2 + 4x - 5$
2 $3\ x^3 - 4x^2 + 7x + 2$
3 $-3\ x^3 + 2x^2 + 7x - 1$
4 $-2\ x^3 + 3x^2 - 7x + 1$
5 $5\ x^4 - 3x^3 + 2x^2 + x - 1$
6 $-7\ x^4 - 6x^3 + x^2 - x + 1$
7 $-2\ 3x^4 - x^3 + x^2 - 3x + 5$
8 $-3\ 2x^4 + 7x^3 - x^2 + 2x + 5$
9 $3\ x^4 - 2x^3 + x - 1$
10 $5\ x^4 - 3x^3 + x^2 - 2$
11 $2\ x^5 + x^4 - 2x^3 - 3x^2 - 7x - 6$
12 $3\ x^5 - x^4 - 8x^3 + 13x^2 - 23x + 6$
13 $-3\ x^5 + 3x^4 + x^3 - 2x^2 - 13x + 6$
14 $-2\ x^5 + 3x^4 + 5x^3 + 6x^2 - 3x - 6$
15 $7\ x^6 - x^5 + 3x^2 + x$
16 $-7\ x^5 + x^4 - 3x^2 + 2x$
17 $-5\ 3x^5 + x^4 - 4x^3 + 1$
18 $-5\ 2x^6 + x^5 - x^3 + 3$

It will now be explained how the factor theorem simplifies the process of finding roots of an equation after one root has been found This will be illustrated with the root -1 of

$$x^6 + 6x^5 + 11x^4 + 14x^3 + x^2 - 35x - 28 = 0 \tag{30}$$

Here $f(x) \equiv x^6 + 6x^5 + 11x^4 + 14x^3 + x^2 - 35x - 28$, $n = 6$, and $c = -1$ Then, by the factor theorem,

$$\begin{aligned} x^6 + 6x^5 + 11x^4 + 14x^3 + x^2 - 35x - 28 \\ \equiv (x + 1)(x^5 + 5x^4 + 6x^3 + 8x^2 - 7x - 28) \end{aligned} \tag{31}$$

Now let r be a root of (30) which is not -1 Since (31) is true for all values of x, a true relation among numbers is obtained by replacing x in (31) by r Thus

$$\begin{aligned} r^6 + 6r^5 + 11r^4 + 14r^3 + r^2 - 35r - 28 \\ = (r + 1)(r^5 + 5r^4 + 6r^3 + 8r^2 - 7r - 28) \end{aligned} \tag{32}$$

Since r is a root of (30), the number on the left-hand side of (32) is indeed zero. Therefore

$$(r + 1)(r^5 + 5r^4 + 6r^3 + 8r^2 - 7r - 28) = 0.$$

Since $r \neq -1$, $r + 1 \neq 0$. Hence

(33) $$r^5 + 5r^4 + 6r^3 + 8r^2 - 7r - 28 = 0.$$

Hence r is a root of the equation

(34) $$x^5 + 5x^4 + 6x^3 + 8x^2 - 7x - 28 = 0.$$

This proves that a root of the particular equation $f(x) = 0$, which is different from the root -1, is indeed a root of the equation $q(x) = 0$. Since $q(x)$ is of lower degree than $f(x)$, the determination of the roots of $f(x) = 0$ should be continued by determining the roots of $q(x) = 0$. Equation (34) is called the depressed equation for $f(x) = 0$ determined by the root -1 of $f(x) = 0$. A different root of $f(x) = 0$ would yield a different depressed equation.

Now $q(x) \equiv x^5 + 5x^4 + 6x^3 + 8x^2 - 7x - 28$. By synthetic substitution it is found that $q(-4) = 0$. Then, by the factor theorem,

(35) $$x^5 + 5x^4 + 6x^3 + 8x^2 - 7x - 28 \equiv (x + 4)(x^4 + x^3 + 2x^2 - 7).$$

Hence

(36) $$x^6 + 6x^5 + 11x^4 + 14x^3 + x^2 - 35x - 28 \equiv (x + 1)(x + 4)(x^4 + x^3 + 2x^2 - 7).$$

Further tests should be made with the function $x^4 + x^3 + 2x^2 - 7$.

It will now be proved for the general polynomial (6) with $n > 0$ that there is a depressed equation determined by a root r_1 of $f(x) = 0$. It is to be noted especially that r_1 is not necessarily an integer, and that, indeed, r_1 may not be real. This is also true of each coefficient in $f(x)$. Since r_1 is a root of $f(x) = 0$, by the factor theorem there is a polynomial $q_1(x)$ of degree $n - 1$ such that

(37) $$f(x) \equiv (x - r_1)q_1(x).$$

The depressed equation determined by the root r_1 of $f(x) = 0$ is $q_1(x) = 0$.

If r_2 is a root of $f(x) = 0$, then $0 = f(r_2) = (r_2 - r_1)q_1(r_2)$. If also $r_2 \neq r_1$, then $q_1(r_2) = 0$, and r_2 is a root of $q_1(x) = 0$. By

the factor theorem applied to $q_1(x)$ and r_2 there is a polynomial $q_2(x)$ of degree $n-2$ such that $q_1(x) \equiv (x - r_2)q_2(x)$ Therefore by substitution in (37)

$$(38) \qquad f(x) \equiv (x - r_1)(x - r_2)q_2(x)$$

It is to be noted especially that $r_2 \neq r_1$ by hypothesis It is also especially to be noted that the coefficient of x^{n-1} in $q_1(x)$ is precisely the coefficient a_0 of x^n in $f(x)$, and similarly that the coefficient of x^{n-2} in $q_2(x)$ is precisely the coefficient of x^{n-1} in $q_1(x)$ and hence is a_0 Continuation of this process shows that theorem 5 is true It can also be proved by induction

THEOREM 5 *If n is a positive integer, if $f(x)$ is a polynomial of degree n if k is an integer such that $1 \leq k \leq n$, and if $r_1, \quad , r_k$ are distinct roots of $f(x) = 0$, then there is a polynomial $q_k(x)$, of degree $n - k$ whose leading coefficient is the leading coefficient a_0 of $f(x)$, such that*

$$(39) \qquad f(x) \equiv (x - r_1)(x - r_2) \cdots (x - r_k)q_k(x)$$

THEOREM 6 *If n is a positive integer and if $f(x)$ is a polynomial in x of degree n, then there are at most n distinct roots of the equation $f(x) = 0$*

Theorem 6 will be proved by showing that if $r, r_1, \quad , r_n$ are $n+1$ distinct roots, then there is a contradiction By theorem 5 $f(x) \equiv a_0(x - r_1) \quad (x - r_n)$ Since r is a root of $f(x) = 0$, therefore $0 = f(r) = a_0(r - r_1) \quad (r - r_n)$ But by hypothesis $r \neq r_1, \quad , r \neq r_n$ Also $a_0 \neq 0$ by the definition of a polynomial of degree n Thus the product on the right is not zero This constitutes a contradiction

It will now be proved that, if $f(x)$ has the notation (6) if $g(x)$ has the notation $b_0x^n + b_1x^{n-1} + \quad + b_{n-1}x + b_n$ and if there are $n+1$ distinct numbers $s_1, \quad , s_{n+1}$ such that $f(s_1) = g(s_1)$, $, f(s_{n+1}) = g(s_{n+1})$, then $b_0 = a_0$, $b_1 = a_1, \quad , b_n = a_n$ The proof uses an auxiliary function $\phi(x)$, which is, by definition, $f(x) - g(x)$ Then

$$(40) \quad \phi(x) \equiv (a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \qquad + (a_{n-1} - b_{n-1})x + (a_n - b_n)$$

Now $\phi(s_1) = 0$, since $\phi(x) \equiv f(x) - g(x)$ and $f(s_1) = g(s_1)$. In this way it is proved that the polynomial equation $\phi(x) = 0$ has $n + 1$ distinct roots $s_1, s_2, \cdots, s_{n+1}$. If any one of the coefficients in (40) were different from zero, there would be a contradiction of theorem 6. Hence $b_0 = a_0$, $b_1 = a_1$, $\cdots$, $b_n = a_n$. This is what is meant by the statement that *the polynomials are term-by-term identical.* The following theorem has thus been proved.

THEOREM 7. *If n is a positive integer, if $f(x)$ and $g(x)$ are two polynomials of degree n, and if $s_1, \cdots, s_{n+1}$ are $n + 1$ distinct numbers such that $f(s_1) = g(s_1)$, $\cdots$, $f(s_{n+1}) = g(s_{n+1})$, then $f(x)$ and $g(x)$ are term-by-term identical.*

PROBLEMS

Find the integral roots of each of the following equations. Find the factorization of the polynomial which is determined by these roots.

1. $x^4 + x^3 + x^2 + 7x - 42 = 0.$
2. $x^4 - 7x^3 + 16x^2 - 28x + 48 = 0.$
3. $x^4 + 7x^3 + x^2 - 63x - 90 = 0.$
4. $x^4 - 2x^3 - 19x^2 + 8x + 60 = 0.$
5. $x^4 - 2x^3 - 9x^2 + 10x + 20 = 0.$
6. $x^4 - 3x^3 - 7x^2 + 9x + 12 = 0.$
7. $x^4 - 5x^3 - 3x^2 + 26x + 8 = 0.$
8. $x^4 - 9x^3 + 21x^2 - 20x + 12 = 0.$
9. $x^4 - 2x^3 - 11x^2 - 16x - 20 = 0.$
10. $x^4 + 2x^3 - 5x^2 + 2x + 24 = 0.$
11. $x^4 + 3x^3 - 6x^2 - 21x - 7 = 0.$
12. $x^4 - x^3 - 6x^2 + 5x + 5 = 0.$
13. $x^5 - x^4 - 13x^3 + 13x^2 + 36x - 36 = 0.$
14. $x^5 - 4x^4 - 4x^3 + 25x^2 - 36 = 0.$
15. $x^5 - 3x^4 - 17x^3 - 6x^2 - 2x + 12 = 0.$
16. $x^5 + 2x^4 - 34x^3 - 85x^2 + 4x + 12 = 0.$

3. Upper and lower bounds for real roots of a real polynomial equation. Some preliminary information about the roots of a real polynomial equation should be obtained before any test is made to determine whether a particular number is a root. Thus, theorem 1 is used if integral roots of a polynomial equation with integral coefficients are under consideration. Some theorems which concern all real roots of a polynomial equation with real coefficients will now be proved.

One very simple fact of this nature is illustrated by the equation $x^3 + 8x^2 + 19x + 12 = 0$. If t is any positive number, then t^3

is also positive Also $8t^2 > 0$, $19t > 0$, and $12 > 0$ By addition of these inequalities $t^3 + 8t^2 + 19t + 12 > 0$ Therefore, if $t > 0$, then t is not a root of the equation $x^3 + 8x^2 + 19x + 12 = 0$ In general, if $f(x)$ is a real polynomial (6) with $n > 0$, if $a_0 > 0$, $a_1 \geqq 0$, , $a_n \geqq 0$, and if t is a positive number, then t is not a root of the equation $f(x) = 0$ Also, if r is a real root of the equation $f(x) = 0$, then $r \leqq 0$

Another simple fact of this nature is illustrated by the equation $-x^3 - 8x^2 - 19x - 12 = 0$ Now, a real number r is a root of this equation if and only if it is a root of $+x^3 + 8x^2 + 19x + 12 = 0$ In general if $f(x)$ is a real polynomial (6) with $n > 0$ and if $a_0 < 0$ $a_1 \leqq 0$, $a_n \leqq 0$ then $-f(x)$ is a real polynomial of degree n such that each of its coefficients is positive or zero By the preceding argument it follows that, if there is any real root r of $f(x) = 0$ then $r \leqq 0$ This completes the proof of the following theorem

THEOREM 8 *If n is a positive integer and $f(x)$ is a polynomial in x of degree n such that either each coefficient is positive or zero or each coefficient is negative or zero, then $f(x) = 0$ has no positive roots*

A simple but important fact which is of use in the solution of equations will now be proved As an illustration it may be verified that, if t is a root of the equation $x^3 - 2x^2 - 5x + 6 = 0$, that is, if $t^3 - 2t^2 - 5t + 6 = 0$ then $-(-t)^3 - 2(-t)^2 + 5(-t) + 6 = 0$ Therefore $-t$ is a root of the equation $-y^3 - 2y^2 + 5y + 6 = 0$ This equation in y is also obtained if x in the original equation is replaced by $-y$ Thus if $f(x) \equiv x^3 - 2x^2 - 5x + 6$ and $g(y) = -y^3 - 2y^2 + 5y + 6$ then $f(-y) \equiv (-y)^3 - 2(-y)^2 - 5(-y) + 6 = g(y)$ In general if x in the function $f(x)$ is replaced by $-y$ and if the result is designated by $g(y)$, it is said that $f(x) = g(y)$ *under the transformation* $x = -y$ If t is a value of x such that $f(t) = 0$, then $g(-t) = 0$ Therefore, if t is a root of $f(x) = 0$, then $-t$ is a root of $g(y) = 0$ This completes the proof of the following theorem

THEOREM 9 *If $g(y)$ designates the result of replacing x by $-y$ in the function $f(x)$ then the roots of $g(y) = 0$ are the negatives of the roots of $f(x) = 0$*

An illustration of the use of theorem 9 in the solution of equations is afforded by the equation $x^3 - 6x^2 + 9x - 6 = 0$ If x is

replaced by $-y$, the equation $-y^3 - 6y^2 - 9y - 6 = 0$ results. Now, by theorem 8, there are no positive roots of this equation in y. Therefore there are no negative roots of $x^3 - 6x^2 + 9x - 6 = 0$. In general, let n be a positive integer and $f(x)$ a polynomial in x of degree n, and let $g(y)$ be the polynomial in y such that $f(x) \equiv g(y)$ under the transformation $x = -y$. If each coefficient in $g(y)$ is positive or zero, or if each coefficient in $g(y)$ is negative or zero, then $g(y) = 0$ has no positive roots, by theorem 8. Hence, by theorem 9, $f(x) = 0$ has no negative roots. This completes the proof of the following theorem.

THEOREM 10. *If n is a positive integer, if $f(x)$ is a polynomial in x of degree n, if $g(y)$ is the polynomial in y such that $f(x) \equiv g(y)$ under the transformation $x = -y$, and if each coefficient in $g(y)$ is positive or zero, or if each coefficient in $g(y)$ is negative or zero, then $f(x) = 0$ has no negative roots.*

If $f(x)$ satisfies the hypothesis of theorem 8, then it is known that there are no positive roots of $f(x) = 0$. If $f(x)$ does not satisfy the hypothesis of theorem 8, then $f(x) = 0$ may or may not have positive roots. One method of obtaining information about whatever real roots it may have will now be explained.

Before theorem 11 is proved, it will be used to obtain information about the real roots of

$$3x^3 + 11x^2 - 2x - 24 = 0. \tag{41}$$

For this equation $n = 3$ and $a_0 = 3$. The terms which have negative coefficients involve x^1 and x^0. The greater of these exponents is 1. By definition h is the greatest exponent of the powers of x having negative coefficients. Hence $h = 1$ and $n - h = 2$. Finally, the absolute values of all the negative coefficients are 2 and 24. The greater of these is designated by G; hence $G = 24$. The theorem states that, if r is a root of (41), then $r < 1 + \sqrt[n-h]{G/a_0}$. Hence $r < 1 + \sqrt{24/3} = 1 + 2\sqrt{2}$. Now $2\sqrt{2} < 3$. Hence $r < 4$. If this result of theorem 11 about positive roots of $3x^3 + 11x^2 - 2x - 24 = 0$ is combined with the result of theorem 1 about integral roots of this equation, the process of finding the positive integral roots of this equation is greatly simplified. Thus, by theorem 1, the only possible positive integral roots are 1, 2, 3, 4, 6, 8, 12, 24. By theorems 1 and 11 the only possible positive integral roots are 1, 2, 3.

Theorem 11 will now be used to obtain information about whatever negative roots equation (41) may have By theorem 9 the negative roots of (41) give the positive roots of $-3y^3 + 11y^2 + 2y - 24 = 0$, that is, of $3y^3 - 11y^2 - 2y + 24 = 0$ Here $n = 3$, $a_0 = 3$, $h = 2$, $G = 11$ By theorem 11 a positive root of this equation in y is less than $1 + \sqrt[3-2]{11/3} = 1 + (11/3) = 14/3$ Hence, if t is a negative root of (41), then $t > -14/3$ If this result of theorem 11 about negative roots is combined with the result of theorem 1 about integral roots, it is found that the only possible negative integral roots are $-1, -2, -3, -4$

Before theorem 11 is proved a simplification of notation will be explained This was illustrated by the equations just preceding theorem 8 In general, the polynomial equations $g(x) = 0$ and $-g(x) = 0$ have the same roots In one of these equations the coefficient of the term which involves the largest exponent of x is positive whereas in the other equation this coefficient is negative Whichever of these equations has this coefficient positive can be used in finding the roots Therefore the notation may be assigned so that

$$(42) \quad f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \quad a_0 > 0$$

THEOREM 11 *Let $f(x)$ designate the polynomial $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, in which n is positive and the coefficients are real numbers Let a_0 be positive and at least one of the coefficients $a_1, \ldots, a_n$ be negative Define h as the greatest exponent of all the powers of x which have negative coefficients Define G as the greatest of the absolute values of all the coefficients which are negative If r is a positive root of $f(x) = 0$, then $r < 1 + \sqrt[n-h]{G/a_0}$*

This theorem will be proved by showing that, if s is any positive number such that $s \geq 1 + \sqrt[n-h]{G/a_0}$, then $f(s) > 0$, and s is not a root of $f(x) = 0$ It will follow that, if r is a positive number such that $f(r) = 0$, then $r < 1 + \sqrt[n-h]{G/a_0}$

The term in $f(x)$ which involves x^h is the term $a_{n-h}x^h$ It is possible that $h = 0$, that is that $a_{n-h}x^h$ is in fact a_n Again, it is possible that $h = n - 1$, that is, that the term a_1x^{n-1} is in fact the term $a_{n-h}x^h$ Nevertheless, $f(x)$ is written in the form

$$(43) \quad f(x) \equiv a_0x^n + \cdots + a_{n-h-1}x^{h+1} + a_{n-h}x^h + a_{n-h+1}x^{h-1} + \cdots + a_{n-1}x + a_n,$$

with the understanding that these possibilities are not excluded by the notation. The right-hand side may terminate with the term $a_{n-h}x^h$, or there may be no terms between a_0x^n and $a_{n-h}x^h$.

It will now be proved that, if $s \geqq 1 + \sqrt[n-h]{G/a_0}$, then

$$(44) \qquad a_0s^n + \cdots + a_{n-h-1}s^{h+1} \geqq a_0s^n.$$

If $h = n - 1$, then the left-hand side of (44) means merely a_0s^n, by the preceding understanding regarding the notation (43). Also $a_0s^n = a_0s^n$. Therefore $a_0s^n \geqq a_0s^n$. Therefore (44) is true if $h = n - 1$. The proof that (44) is true if $h < n - 1$ uses the facts that $a_1 \geqq 0, \cdots, a_{n-h-1} \geqq 0$, which are implied by the definition of h and $h < n - 1$, and the fact that $s > 0$, which is implied by the hypothesis that $s \geqq 1 + \sqrt[n-h]{G/a_0}$. Thus $a_1 \geqq 0$ and $s > 0$ imply that $a_1s^{n-1} \geqq 0$. Similarly $a_2s^{n-2} \geqq 0, \cdots, a_{n-h-1}s^{h+1} \geqq 0$. If these inequalities and the equality $a_0s^n = a_0s^n$ are added, the result is (44). By the definitions of h and G, it is true that $-a_{n-h} \leqq G$ and $a_{n-h} \geqq -G$. Hence $a_{n-h}s^h \geqq -Gs^h$. If this equality is added to the inequality (44), the result is the inequality

$$(45) \qquad a_0s^n + \cdots + a_{n-h-1}s^{h+1} + a_{n-h}s^h \geqq a_0s^n - Gs^h.$$

It will now be proved that, if the left-hand side of (45) contains all the terms of $f(s)$, that is, if $h = 0$, then $f(s) > 0$. If $h = 0$, then (45) becomes

$$(46) \qquad f(s) \geqq a_0s^n - G.$$

Now, by the hypothesis that $s \geqq 1 + \sqrt[n-0]{G/a_0}$, it follows that $(s - 1)^n \geqq G/a_0$. But $s > s - 1 > 0$. Hence $a_0s^n > a_0(s - 1)^n \geqq G$. Hence $a_0s^n - G > 0$. Hence it follows by (46) that $f(s) > 0$.

It remains to prove that $f(s) > 0$ if the left-hand side of (45) does not contain all the terms of $f(s)$, that is, if $h > 0$. If i is the subscript of the coefficient of a term which is in $f(s)$ but which does not appear in the left-hand side of (45), then $n - h < i \leqq n$. There are two possibilities: either $a_i \geqq 0$, or $a_i < 0$. If $a_i \geqq 0$, then $a_i > -G$, since $-G$ is negative. Then $a_is^{n-i} \geqq -Gs^{n-i}$. The inequality $a_is^{n-i} \geqq -Gs^{n-i}$ will now be proved if $a_i < 0$. Thus, by the definition of G, $0 < -a_i \leqq G$. Hence $-a_is^{n-i} \leqq Gs^{n-i}$. Hence $a_is^{n-i} \geqq -Gs^{n-i}$. Thus it has been proved that

$$(47) \quad a_{n-h+1}s^{h-1} \geqq -Gs^{h-1}, \cdots, a_{n-1}s \geqq -Gs, a_n \geqq -G.$$

If these inequalities are added to the inequality (45), then the result is the inequality

$$(48)\quad a_0s^n + \cdots + a_{n-h-1}s^{h+1} + a_{n-h}s^h + a_{n-h+1}s^{h-1} + \cdots + a_{n-1}s + a_n \geqq a_0s^n - Gs^h - Gs^{h-1} - \cdots - Gs - G$$

Hence, by (43), it follows that

$$(49)\qquad f(s) \geqq a_0s^n - G(s^h + s^{h+1} + \cdots + s + 1)$$

Now it is known that $s^{h+1} - 1 = (s-1)(s^h + s^{h-1} + \cdots + s + 1)$ Since $s > 1$, it follows that $s - 1 \neq 0$ and hence that

$$(50)\qquad f(s) \geqq a_0s^n - \frac{G(s^{h+1} - 1)}{s-1}$$

Also, if the right-hand side of (50) is expressed as a single fraction, it becomes $[a_0s^n(s-1) - G(s^{h+1} - 1)]/(s-1)$ This expression is equal to $\{s^{h+1}[a_0s^{n-h-1}(s-1) - G]/(s-1)\} + G/(s-1)$ Since $G > 0$ and $s - 1 > 0$, it is true that $G/(s-1) > 0$ Therefore, by (50),

$$(51)\qquad f(s) > \frac{s^{h+1}[a_0s^{n-h-1}(s-1) - G]}{s-1}$$

Now $s > s - 1$ Hence $s^{n-h-1} > (s-1)^{n-h-1}$ Hence $a_0s^{n-h-1}(s-1) > a_0(s-1)^{n-h}$, and therefore $a_0s^{n-h-1}(s-1) - G > a_0(s-1)^{n-h} - G$ Hence $s^{h+1}[a_0s^{n-h-1}(s-1) - G]/(s-1) > s^{h+1}[a_0(s-1)^{n-h} - G]/(s-1)$ Hence, by (51), it is true that

$$(52)\qquad f(s) > \frac{s^{h+1}[a_0(s-1)^{n-h} - G]}{s-1}$$

Now, by the hypothesis that $s \geqq 1 + \sqrt[n-h]{G/a_0}$, it follows that $(s-1)^{n-h} \geqq G/a_0$ and hence that $a_0(s-1)^{n-h} - G > 0$ The other terms, s^{h+1} and $s - 1$, on the right-hand side of (52) also are positive Therefore $f(s) > 0$ This completes the proof of theorem 11

If n is a positive integer and $f(x)$ is a polynomial in x of degree n whose coefficients are real numbers, then the statement that *a real number t is an upper bound for the real roots of $f(x) = 0$* means that, if r is a real root of $f(x) = 0$, then $r < t$ If an equation satisfies the hypotheses of theorem 11, the number $1 + \sqrt[n-h]{G/a_0}$

is an upper bound for its real roots. The statement that *a real number b is a lower bound for the real roots of $f(x) = 0$* means that, if r is a real root of $f(x) = 0$, then $b < r$.

If $f(x)$ is a real polynomial whose coefficients are not all of one sign, and if $g(y)$ is related to $f(x)$ as described in theorem 10, then an upper bound t for the positive roots of $g(y) = 0$ can be determined by theorem 11. The negative of t is a lower bound for the negative roots of $f(x) = 0$.

PROBLEMS

Find an upper bound and a lower bound for the real roots of each of the following equations.

1. $x^5 + 3x^4 + x^3 + 2x^2 + x + 1 = 0.$
2. $x^5 + 2x^4 - x^3 - 7x^2 + x - 2 = 0.$
3. $x^5 + 7x^4 - x^3 + x^2 - 5x - 1 = 0.$
4. $x^5 + 2x^4 + x^3 + 3x^2 + 5x + 2 = 0.$
5. $x^5 + 2x^4 + x^3 - 3x^2 + x - 11 = 0.$
6. $x^5 + x^4 + 2x^3 - x^2 + 3x - 5 = 0.$
7. $x^5 + x^3 - 2x^2 + 13x + 1 = 0.$
8. $x^5 + x^4 - x^2 + x - 2 = 0.$
9. $x^5 + x^2 - x - 7 = 0.$
10. $x^5 - x^2 - 3x + 1 = 0.$
11. $x^5 - x^4 + 2x^3 - x^2 - 7x = 0.$
12. $x^5 + 5x^4 - x^3 + 2x^2 - 11x = 0.$
13. $-2x^5 + x^4 - x^3 + 3x^2 + x - 2 = 0.$
14. $-3x^5 - 2x^4 + x^3 + 5x^2 - 2x + 1 = 0.$
15. $x^5 - 3x^4 + 2x^3 - x^2 + 7x - 1 = 0.$
16. $x^5 - 5x^4 + x^3 - 2x^2 + 3x - 2 = 0.$

4. Rational roots of a polynomial equation whose coefficients are integers. The statement that *the integers r and s are coprime* means that, if t is an integer which is a factor of r and a factor of s, then t is 1 or -1. It is also said that *r and s are relatively prime*. *A rational number* is the quotient of two integers. The rational number $4/8$ equals the rational number $1/2$. In general, the notation c/d for a rational number may be chosen so that the integers c and d are coprime. It is said that *c/d is in lowest terms* if c and d are coprime. An integer c is a rational number since $c = c/1$. If a rational number c/d is in lowest terms and is an integer, then $d = 1$ or $d = -1$.

The proof of theorem 12 and the use of theorem 12 to obtain information about the rational roots of a polynomial equation

with integral coefficients will now be illustrated If c and d are coprime integers, and if c/d is a root of

(53) $$96x^3 + 4x^2 - 31x + 6 = 0,$$

then $96(c/d)^3 + 4(c/d)^2 - 31(c/d) + 6 = 0$ Hence $96c^3 + 4c^2d - 31cd^2 + 6d^3 = 0$ This equation can be written in the form

(54) $$96c^3 = d(-4c^2 + 31cd - 6d^2)$$

and also in the form

(55) $$6d^3 = c(31d^2 - 4cd - 96c^2)$$

In (54) the number $-4c^2 + 31cd - 6d^2$ is an integer, since c and d are integers Therefore (54) states that the integer d divides the integer $96c^3$ By hypothesis the integer d has no factor which is greater than 1 in common with c Therefore d is a factor of 96 Again by (55), c is a factor of 6 For example, $1/96$ and $-2/3$ are possible rational roots of (53), and $1/5$ is not a rational root

THEOREM 12 *If $f(x)$ is a polynomial (6) whose coefficients are integers if c and d are relatively prime integers and if c/d is a root of $f(x) = 0$, then c is a factor of the constant term a_n in $f(x)$, and d is a factor of the leading coefficient a_0 in $f(x)$*

PROOF If $f(x)$ is the polynomial (6) if the coefficients of (6) are integers and if c and d are relatively prime integers such that c/d is a root of $f(x) = 0$, then

$$a_0\left(\frac{c}{d}\right)^n + a_1\left(\frac{c}{d}\right)^{n-1} + \quad + a_{n-1}\left(\frac{c}{d}\right) + a_n = 0,$$

and hence $a_0c^n + a_1c^{n-1}d + \quad + a_{n-1}cd^{n-1} + a_nd^n = 0$ Hence

(56) $$c(a_0c^{n-1} + a_1c^{n-2}d + \quad + a_{n-1}d^{n-1}) = a_n(-d^n),$$

and

(57) $$d(a_nd^{n-1} + a_{n-1}cd^{n-2} + \quad + a_1c^{n-1}) = a_0(-c^n)$$

Since c and d are relatively prime integers, it follows from (56) that c is a factor of a_n, and it follows from (57) that d is a factor of a_0

THEOREM 13 *If $f(x)$ is a polynomial (6) whose coefficients are integers, and if the leading coefficient a_0 is 1, then a rational root of $f(x) = 0$ is an integer*

PROOF. By theorem 12 the integer d is a factor of a_0. Therefore $d = 1$ or $d = -1$, and c/d is an integer.

One method of determining all the rational roots of (53) would be to test all the possible fractions which, by theorem 12, were possible roots. The details are impracticable because there are so many of these fractions and because synthetic substitution with a fraction is intricate. On the other hand, the following method is effective and involves few fractions. If

$$(58) \qquad y = 96x,$$

then (53) becomes $96(y/96)^3 + 4(y/96)^2 - 31(y/96) + 6 = 0$. In this equation fractions will be cleared by multiplication by $(96)^2$. The result is

$$(59) \qquad y^3 + 4y^2 - 31 \cdot 96y + 6(96)^2 = 0.$$

A more simple method of obtaining (59) from (53) by (58) is to multiply (53) by $(96)^2$ before (58) is used. Thus, (53) is equivalent to the equation $96^3x^3 + 96^2 \cdot 4x^2 - 31 \cdot 96^2 x + 6 \cdot 96^2 = 0$, and hence to $(96x)^3 + 4(96x)^2 - 96 \cdot 31(96x) + 96^2 \cdot 6 = 0$. Hence by (58) the equation (59) is obtained.

It will now be explained precisely how the new equation (59) is used in finding the rational roots of (53). It was proved earlier that, if c/d is a rational root of (53) and in lowest terms, then the integer d divides 96. Therefore, by (58), the corresponding value of y, being $96(c/d)$, is an integer. This integer is a root of (59), because (53) becomes (59) under the transformation (58). Therefore each rational root of (53) determines, by (58), an integral root of (59). The converse of this statement will now be proved. Thus, if s is an integral root of (59), then, by (58), the corresponding value of x is $s/96$. This rational number is a root of (53) because (59) becomes (53) under the transformation (58). Therefore all rational roots of (53) are obtained by finding all integral roots of (59) and using (58). Upper and lower bounds for the real roots of (59) would be found by theorem 11. Then the divisors of the constant term of (59) which are between these bounds are the only possible integral roots of (59).

It frequently happens in practice that a smaller multiple of x is equally effective and yields a new equation with smaller coefficients. The method of finding the smallest multiple of x which is effective will now be illustrated. The leading coefficient 96 in

equation (53) equals $2^5\ 3$ If (53) is multiplied by $2\ 3^2$ the result is

$$(60) \qquad 2^6\ 3^3x^3 + 2\ 3^2\ 4x^2 - 2\ 3^2\ 31x + 2\ 3^2\ 6 = 0$$

In this equation the leading term is $(2^2\ 3x)^3$ This suggests the substitution $z = 2^2\ 3x$ Then the second term would be $(1/2)z^2$ The resulting equation could not be used because in the theorems which have been proved there is the hypothesis that the coefficients are integers This indicates that (60) should be multiplied by 2^3 The result is

$$(61) \qquad 2^9\ 3^3x^3 + 2^6\ 3^2x^2 - 2^4\ 3^2\ 31x + 2^4\ 3^2\ 6 = 0$$

This can be rewritten in the form

$$(62) \qquad (2^3\ 3x)^3 + (2^3\ 3x)^2 - 186(2^3\ 3x) + 864 = 0$$

Now the substitution

$$(63) \qquad z = 2^3\ 3x$$

leads to the equation

$$(64) \qquad z^3 + z^2 - 186z + 864 = 0$$

By (63), if c/d is a value of x which satisfies (53), then $24(c/d)$ is the corresponding value of z and satisfies (64) By theorem 13 each rational root of (64) is an integer Therefore each rational root of (53) determines, by (63), an integral root of (64) Conversely, each integral root of (64) determines by (63), a rational root of (53) Hence all the rational roots of (53) are obtained from the integral roots of (64) by division by 24

The integral roots of (64) will now be found By theorem 11, the number 15 is an upper bound to the roots of (64) Also, the constant term 864 has the factorization $2^5\ 3^3$ Hence, by theorem 1, if there are any positive integral roots of (64), they are in the list 1, 2, 3, 4, 6, 8, 9, 12 Again, by theorem 11, the number -865 is a lower bound to the roots of (64) Hence, by theorem 1, if there are any negative integral roots of (64), they are in the list which is formed by -1, -2, -4, -8, -16, -32 and the multiples of these six negative integers by 3, by 9, and by 27 Hence the complete list of possible integral roots of (64) contains thirty-two entries

A method will now be explained by which it can be proved, more easily than by synthetic substitution, that many of these possible roots are not roots. If the function $z^3 + z^2 - 186z + 864$ is designated by $h(z)$, then $h(3) = 342$. Thus 3 is not a root of (64). Now $342 = 2 \cdot 3^2 \cdot 19$. The fact that there are so few prime numbers which divide $h(3)$ means that 342 is a very useful value of $h(z)$ in the following process. *First*, if k is an *integer which is* a root of (64), then by theorem 3 there is a polynomial $q(z)$ such that $h(z) \equiv (z - k)q(z)$. Also, by the method of proof of theorem 2, it is true that the coefficients of $q(z)$ are integers, since the coefficients of $h(z)$ are integers and since k is an integer. If z is replaced by 3, the identity yields the true relation $h(3) = (3 - k)q(3)$ between integers. This equation states that the integer $3 - k$ divides the integer 342, since $q(3)$ is an integer. Thus it has been proved that, if k is an integer which is a root of (64), then $3 - k$ divides $2 \cdot 3^2 \cdot 19$. This statement implies that, *if* $3 - k$ *does not divide* $2 \cdot 3^2 \cdot 19$, *then* k *is not a root of* (64). This fact will now be used to prove that many of the thirty-two possible integral roots of (64) are not roots. The results will be tabulated. The third line of the table has the entry "No" if $3 - k$ does not divide $2 \cdot 3^2 \cdot 19$.

k	1	2	4	6	8	9	12	−1	−2	−4	−8
$3 - k$	2	1	−1	−3	−5	−6	−9	4	5	7	11
					No			No	No	No	No

k	−16	−32	−3	−6	−12	−24	−48	−96	−9	−18
$3 - k$	19	35	6	9	15	27	51	99	12	21
		No			No	No	No	No	No	No

k	−36	−72	−144	−288	−27	−54	−108	−216	−432	−864
$3 - k$	39	75	147	291	30	57	111	219	435	867
	No	No	No	No	No		No	No	No	No

Hence the only remaining possible integral roots of (64) are the integers 1, 2, 4, 6, 9, 12, −16, −3, −6, −54. Now $h(4) = 200 = 2^3 \cdot 5^2$. Thus 4 is not a root of (64) and a new table is con-

k	1	2	6	9	12	−16	−3	−6	−54
$4 - k$	3	2	−2	−5	−8	20	7	10	58
	No						No		No

structed. It shows that several of the possible integral roots are not roots. By synthetic substitution $h(2) \neq 0$, and $h(6) = 0$. Also $h(z) \equiv (z - 6)(z^2 + 7z - 144) \equiv (z - 6)(z - 9)(z + 16)$.

Therefore 6, 9, -16 are the roots of (64) By (63) the roots of (53) are $1/4$, $3/8$, $-2/3$

It is to be noted that in this process $h(3)$ and $h(4)$ were used and that 3 and 4 were possible integral roots of (64) However, in this process $h(t)$ may be used even if t is not one of the possible integral roots of (64) This fact is proved in the course of the following general proof

If $f(x)$ is a polynomial in x and k is a root of $f(x) = 0$, then there is a polynomial $q(x)$ such that $f(x) \equiv (x - k)q(x)$ If also k and the coefficients of $f(x)$ are integers, then the coefficients of $q(x)$ are integers, since the only operations in the identities are multiplication, addition and subtraction This fact can also be proved by induction Now, if t is any integer such that $f(t) \neq 0$, then $f(t) = (t - k)q(t)$, *and the integer* $t - k$ *is a factor of the integer* $f(t)$ It follows that if $f(x)$ is a polynomial in x of positive degree, whose coefficients are integers if t is an integer such that $f(t) \neq 0$ and if k is an integer such that $t - k$ is not a factor of $f(t)$, then k is not a root of $f(x) = 0$ This completes the proof of the following theorem

THEOREM 14 *Let $f(x)$ be a polynomial in x of positive degree Let the coefficients of $f(x)$ be integers and t be an integer such that $f(t) \neq 0$ If k is an integer such that $t - k$ is not a factor of $f(t)$, then k is not a root of $f(x) = 0$*

PROBLEMS

Find all the rational roots of each of the following equations

1 $6x^4 - 13x^3 - 63x^2 + 82x - 24 = 0$
2 $6x^4 - 31x^3 - 33x^2 + 16x + 12 = 0$
3 $20x^4 - 7x^3 - 46x^2 + 14x + 12 = 0$
4 $21x^4 + 22x^3 - 71x^2 - 66x + 24 = 0$
5 $6x^4 - 5x^3 + 14x^2 - 15x - 12 = 0$
6 $12x^4 - 5x^3 + 22x^2 - 10x - 4 = 0$
7 $9x^4 - 56x^3 + 57x^2 + 98x - 24 = 0$
8 $9x^4 - 31x^3 - 42x^2 + 96x - 32 = 0$
9 $x^4 - 2x^3 - 2x^2 + 13x - 12 = 0$
10 $x^5 - 8x^4 + 11x^3 + 2x^2 + 18x + 36 = 0$
11 $x^5 + 5x^4 - 5x^3 + 3x^2 - 24x - 36 = 0$
12 $x^4 - 7x^3 + 21x^2 - 28x + 18 = 0$
13 $6x^5 - 11x^4 + 21x^3 - 42x^2 - 12x + 8 = 0$
14 $x^4 - 9x^3 + 22x^2 - 46x + 12 = 0$
15 $x^4 - 6x^3 + 10x^2 - 17x + 6 = 0$
16 $6x^5 + 7x^4 + 23x^3 + 26x^2 - 4x - 8 = 0$

5. Multiple roots. An important method in the solution of polynomial equations will now be illustrated by means of the equation

$$x^4 + 5x^3 + 5x^2 + 3x + 18 = 0. \tag{65}$$

By theorem 13 the rational roots of (65) are indeed integers. By theorem 8 there are no positive roots of (65). By theorem 9 the negative roots of (65) give the positive roots of $y^4 - 5y^3 + 5y^2 - 3y + 18 = 0$. By theorem 11, the number 6 is an upper bound to the positive roots of this equation. Hence -6 is a lower bound to the roots of (65). Hence -1, -2, -3 are the only possible rational roots of (65). By synthetic substitution it is found that -1 and -2 are not roots of (65) and that -3 is a root of (65). Also

$$x^4 + 5x^3 + 5x^2 + 3x + 18 \equiv (x + 3)(x^3 + 2x^2 - x + 6). \tag{66}$$

It is not correct to conclude that there are no rational roots of the depressed equation

$$x^3 + 2x^2 - x + 6 = 0. \tag{67}$$

The fact is that -3 is a root of (67), and that $x^3 + 2x^2 - x + 6 \equiv (x + 3)(x^2 - x + 2)$. Hence

$$x^4 + 5x^3 + 5x^2 + 3x + 18 \equiv (x + 3)^2(x^2 - x + 2). \tag{68}$$

Now -3 is not a root of $x^2 - x + 2 = 0$. Hence $x + 3$ is not a factor of $x^2 - x + 2$. Hence, by (68), $(x + 3)^2$ is a factor of $x^4 + 5x^3 + 5x^2 + 3x + 18$, but $(x + 3)^3$ is not a factor. This is the meaning of the statement that -3 is a root of multiplicity 2 of $x^4 + 5x^3 + 5x^2 + 3x + 18 = 0$.

In general, if $f(x)$ is the polynomial (6) in which the coefficients are complex numbers, and if r is a root of $f(x) = 0$, then by theorem 3 there is a polynomial $q(x)$ such that $f(x) \equiv (x - r)q(x)$. Therefore there is a positive integer m such that $(x - r)^m$ is a factor of $f(x)$ and $(x - r)^{m+1}$ is not a factor of $f(x)$. This integer m is, by definition, *the multiplicity of the root r of the equation $f(x) = 0$.*

If r_1 and r_2 are distinct roots of $f(x) = 0$, then a factorization of $f(x)$ is given in theorem 5. In this factorization the multiplicities of r_1 and r_2 for $f(x) = 0$ do not appear. A factorization in which the multiplicity m_1 of r_1 and the multiplicity m_2 of r_2

appear will now be explained By the definition of m_1 there is a polynomial $q_1(x)$ such that

$$(69) \qquad f(x) \equiv (x - r_1)^{m_1} q_1(x)$$

Hence $f(r_2) = (r_2 - r_1)^{m_1} q_1(r_2)$, and then $0 = (r_2 - r_1)^{m} q_1(r_2)$ Also $r_1 \neq r_2$ by hypothesis Therefore r_2 is a root of $q_1(x) = 0$, and there is a positive integer which is the multiplicity of r_2 for the equation $q_1(x) = 0$ If this multiplicity is designated by n_2, then, by the definition, there is a polynomial $q_2(x)$ such that $q_1(x) \equiv (x - r_2)^{n_2} q_2(x)$ and that $x - r_2$ is not a factor of $q_2(x)$ Hence, by (69), $f(x) \equiv (x - r_1)^{m_1}(x - r_2)^{n_2} q_2(x)$ If $(x - r_1)^{m_1} q_2(x)$ is designated by $q_3(x)$, then $f(x) \equiv (x - r_2)^{n_2} q_3(x)$

It will now be proved that $x - r_2$ does not divide $q_3(x)$, by showing that if there were a polynomial $q_4(x)$ such that $q_3(x) \equiv (x - r_2) q_4(x)$ then there would be a contradiction Thus, if the two expressions for $q_3(x)$ are equated, the result is $(x - r_1)^{m_1} q_2(x) \equiv (x - r_2) q_4(x)$ If x is replaced by the number r_2, this identity gives the true relation $(r_2 - r_1)^{m_1} q_2(r_2) = (r_2 - r_2) q_4(r_2)$ between numbers Hence $(r_2 - r_1)^{m_1} q_2(r_2) = 0$ On the other hand, it will now be proved that $r_2 - r_1 \neq 0$ and $q_2(r_2) \neq 0$, and hence that $(r_2 - r_1)^{m} q_2(r_2) \neq 0$ This is the contradiction that was mentioned By hypothesis $r_2 \neq r_1$ Therefore $r_2 - r_1 \neq 0$ Again, if $q_2(r_2)$ were zero, then r_2 would be a root of $q_2(x) = 0$ Hence, by the factor theorem, it would be true that $x - r_2$ is a factor of $q_2(x)$ But by the definition of $q_2(x)$, it is known that $x - r_2$ is not a factor of $q_2(x)$ Therefore $q_2(r_2)$ is not zero

It has been proved that $f(x) \equiv (x - r_2)^{n_2} q_3(x)$ and that $x - r_2$ does not divide $q_3(x)$ This means that n_2 is indeed the multiplicity of r_2 for $f(x) = 0$ It is especially to be noted that n_2 was, by definition, the multiplicity of r_2 for $q_1(x) = 0$, and that the multiplicity of r_2 for $f(x) = 0$ was designated by m_2 Therefore $n_2 = m_2$ It was also proved that $f(x) \equiv (x - r_1)^{m} (x - r_2)^{n_2} q_2(x)$ Therefore, finally, it has been proved that, if r_1 and r_2 are distinct roots of $f(x) = 0$, of multiplicities m_1 and m_2 respectively, then there is a polynomial $q_2(x)$ such that $f(x) \equiv (x - r_1)^{m_1}(x - r_2)^{m_2} q_2(x)$

In general, if k is an integer greater than 1, and if $r_1, r_2, \cdots, r_k$ are k distinct roots of $f(x) = 0$ then $k \leq n$, by theorem 6 If $m_1, m_2, \cdots, m_k$ are the multiplicities of $r_1, r_2, \cdots, r_k$ respectively

for $f(x) = 0$, then it can be proved by induction that $m_1 + m_2 + \cdots + m_k \leqq n$ and there is a polynomial $q_k(x)$ such that $f(x) \equiv (x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_k)^{m_k} q_k(x)$. In (69) the leading coefficient of $q_1(x)$ is the leading coefficient a_0 of $f(x)$. In general, the leading coefficient of $q_k(x)$ is a_0. Hence, if $m_1 + m_2 + \cdots + m_k = n$, then $q_k(x)$ is a_0. This completes the proof of the following theorem.

Theorem 15. *If $f(x)$ is the polynomial $a_0x^n + \cdots + a_{n-1}x + a_n$, if $r_1, \cdots, r_k$ are distinct roots of $f(x) = 0$, and if the multiplicities of $r_1, \cdots, r_k$ for $f(x) = 0$ are $m_1, \cdots, m_k$ respectively, then $m_1 + \cdots + m_k \leqq n$, and there is a polynomial $Q(x)$ such that $f(x) \equiv (x - r_1)^{m_1} \cdots (x - r_k)^{m_k} Q(x)$. The leading coefficient of $Q(x)$ is a_0. If $m_1 + \cdots + m_k = n$, then $Q(x)$ is the constant a_0.*

It is to be noted especially that, if r_1 is a root of multiplicity m_1 for $f(x) = 0$, then, by (69), the equation $f(x) = 0$ can be written as $(x - r_1)^{m_1} q_1(x) = 0$, and r_1 is counted as m_1 equal roots. Similarly, if r_1 and r_2 are distinct roots of $f(x) = 0$, of multiplicities m_1 and m_2 respectively, then $f(x) = 0$ can be written as $(x - r_1)^{m_1}(x - r_2)^{m_2} q_2(x) = 0$, and r_1 is counted as m_1 equal roots and r_2 as m_2 equal roots. In general, *each number r which is a root of $f(x) = 0$ has a multiplicity m, and r is counted as m equal roots.* If $m > 1$, then r is called a multiple root of $f(x) = 0$. If $m = 1$, then r is called a simple root of $f(x) = 0$. Therefore the following theorem is implied by theorem 15.

Theorem 16. *If a root of multiplicity m is counted as m roots, then a polynomial equation of degree n has at most n roots.*

It will now be explained how to determine whether a polynomial equation has any multiple roots. If it has a multiple root, a new equation will be found, which is of lower degree than the original equation and which has the same roots as the original equation but no multiple roots.

Let $f(x)$ be the general polynomial (6), and let $f'(x)$ be its first derivative. Let r be a root of $f(x) = 0$, and let m be its multiplicity. It will now be proved that, if $m > 1$, then r is a root of $f'(x) = 0$ of multiplicity $m - 1$, but that, if $m = 1$, then r is not a root of $f'(x) = 0$. By the definition of the multiplicity of a root of an equation, there is a polynomial $q(x)$ such that $f(x) \equiv (x - r)^m q(x)$ and $x - r$ is not a factor of $q(x)$. By the rule for dif-

ferentiating a product, $f'(x) \equiv (x-r)^m q'(x) + m(x-r)^{m-1}q(x)$ Hence $f'(x) \equiv (x-r)^{m-1}[(x-r)q'(x) + mq(x)]$ Therefore $(x-r)^{m-1}$ is a factor of $f'(x)$ It will now be proved that $(x-r)^m$ is not a factor of $f'(x)$ This will follow if it is proved that $x - r$ *is not a factor of* $(x-r)q'(x) + mq(x)$ *This last* statement will now be proved If there is a polynomial $s(x)$ such that $(x-r)s(x) \equiv (x-r)q'(x) + mq(x)$, then it is true that $(x-r)[s(x) - q'(x)] \equiv mq(x)$ Therefore $x - r$ is a factor of $q(x)$ This contradicts the definition of $q(x)$ Thus it has been proved that $(x-r)^{m-1}$ is a factor of $f'(x)$ and $(x-r)^m$ is not a factor of $f(x)$ This completes the proof of the following theorem

THEOREM 17 *Let r be a root of multiplicity m of the polynomial equation $f(x) = 0$ If $m > 1$, then r is a root of $f'(x) = 0$ of multiplicity $m - 1$ If $m = 1$ then r is not a root of $f'(x) = 0$*

By theorem 17 it is known that a multiple root of $f(x) = 0$ is a common root of $f(x) = 0$ and $f(x) = 0$ It will now be proved that, if s is a common root of $f(x) = 0$ and $f'(x) = 0$, and if m_1 is the multiplicity of s for $f(x) = 0$ and m_2 the multiplicity of s for $f(x) = 0$ then $m_2 = m_1 + 1$ Now either $m_2 > 1$, or $m_2 = 1$ If m_2 is 1, it follows by the last sentence in theorem 17 that s is not a root of $f'(x) = 0$ This is a contradiction of the hypothesis that s is a root of $f(x) = 0$ Hence $m_2 > 1$ Then, by theorem 17, s is a root of $f(x) = 0$ of multiplicity $m_2 - 1$ Hence $m_1 = m_2 - 1$ Therefore $m_2 = m_1 + 1$ This completes the proof of the following theorem

THEOREM 18 *If s is a common root of $f(x) = 0$ and $f'(x) = 0$, and if the multiplicity of s for $f(x) = 0$ is m_1, then the multiplicity of s for $f(x) = 0$ is $m_1 + 1$*

PROBLEMS

In each of the following problems let $f(x)$ mean the polynomial in the given equation Solve $f(x) = 0$ and apply theorem 18 to determine any multiple roots which $f(x) = 0$ has.

1 $x^3 + x^2 - 5x + 3 = 0$
2 $x^3 - 3x^2 - 9x - 5 = 0$
3 $x^3 - 6x^2 + 12x - 8 = 0$
4 $x^3 + 9x^2 + 27x + 27 = 0$
5 $x^3 + 2x^2 - 5x - 6 = 0$
6 $x^3 + 6x^2 - x - 30 = 0$

7. $x^3 + x^2 - 16x + 20 = 0.$
8. $x^3 - 2x^2 - 15x + 36 = 0.$
9. $x^4 - 10x^3 + 36x^2 - 54x + 27 = 0.$
10. $x^4 + 3x^3 - 6x^2 - 28x - 24 = 0.$
11. $x^3 - 3x^2 - 6x + 8 = 0$
12. $x^4 + 2x^3 - 3x^2 - 4x + 4 = 0.$
13. $x^4 - 4x^3 - 2x^2 + 12x + 9 = 0.$
14. $x^3 - 5x^2 - 33x - 27 = 0.$
15. $x^3 + 3x^2 - 45x + 2 = 0.$
16. $x^3 - 3x^2 - 9x + 4 = 0.$

It will now be explained how $f'(x)$ is used to determine whether $f(x) = 0$ has multiple roots and the multiplicities of any such roots. If $f(x)$ is a polynomial of positive degree, then there is a unique polynomial, which will be designated by $g(x)$, with the following three properties: (1) the leading coefficient of $g(x)$ is 1; (2) $g(x)$ is a factor of $f(x)$ and a factor of $f'(x)$; (3) if $d(x)$ is a factor of $f(x)$ and a factor of $f'(x)$, then $d(x)$ is a factor of $g(x)$. This polynomial $g(x)$ is called *the greatest common divisor of $f(x)$ and $f'(x)$*. There is no common divisor of $f(x)$ and $f'(x)$ whose degree is greater than the degree of $g(x)$.

A method of finding $g(x)$ will now be illustrated. If $f(x)$ is the polynomial

(70) $$x^6 + 6x^5 + 9x^4 - 12x^3 - 48x^2 - 48x - 16,$$

then

(71) $$f'(x) = 6x^5 + 30x^4 + 36x^3 - 36x^2 - 96x - 48.$$

The largest integer which is a common factor of the coefficients of $f'(x)$ is 6. It will simplify further details if a new notation is introduced. Thus, if

(72) $$F_1(x) \equiv x^5 + 5x^4 + 6x^3 - 6x^2 - 16x - 8,$$

then $f'(x) \equiv 6F_1(x)$. The first step is to find a polynomial $q_1(x)$, and a polynomial $r_1(x)$ of degree lower than the degree of $F_1(x)$, such that

(73) $$f(x) \equiv q_1(x)F_1(x) + r_1(x).$$

Thus it is verified that $f(x) - xF_1(x) \equiv x^5 + 3x^4 - 6x^3 - 32x^2 - 40x - 16$. Hence $f(x) - xF_1(x) - F_1(x) \equiv -2x^4 - 12x^3 - 26x^2 - 24x - 8$. Hence $f(x) \equiv (x + 1)F_1(x) + (-2x^4 - 12x^3 - 26x^2 - 24x - 8)$. Therefore (73) holds with $q_1(x) \equiv x + 1$

and $r_1(x) = -2x^4 - 12x^3 - 26x^2 - 24x - 8$ Since the coefficients of $r_1(x)$ have the common factor 2, the notation $F_2(x) \equiv -x^4 - 6x^3 - 13x^2 - 12x - 4$ is introduced Then

$$f(x) \equiv q_1(x)F_1(x) + 2F_2(x) \tag{74}$$

There is a more practical method of obtaining $q_1(x)$ and $r_1(x)$ Thus, if the usual process of long division is applied to divide $f(x)$ by $F_1(x)$, the polynomials $q_1(x)$ and $r_1(x)$ appear as quotient and remainder respectively

A simplification of the process of long division will now be illustrated It is referred to as *long division by detached coefficients*, since it is a tabulation of the coefficients only in the usual process of long division Thus the following tabulation is sufficient to obtain the identity (73)

$$\begin{array}{r|rrrrrrr}
 & 1 & 1 & & & & & \\
\hline
1\ \ 5\ \ 6\ \ -6\ \ -16\ \ -8 & 1 & 6 & 9 & -12 & -48 & -48 & -16 \\
 & 1 & 5 & 6 & -6 & -16 & -8 & \\
\cline{2-8}
 & & 1 & 3 & -6 & -32 & -40 & -16 \\
 & & 1 & 5 & 6 & -6 & -16 & -8 \\
\cline{3-8}
 & & & -2 & -12 & -26 & -24 & -8
\end{array}$$

The second step in the process of finding $g(x)$ is the determination of a polynomial $q_2(x)$, and a polynomial $r_2(x)$ of degree lower than the degree of $F_2(x)$ such that

$$F_1(x) \equiv q_2(x)F_2(x) + r_2(x) \tag{75}$$

Thus $q_2(x) \equiv -x + 1$ and $r_2(x) = -x^3 - 5x^2 - 8x - 4$ For uniformity the notation

$$F_3(x) \equiv -x^3 - 5x^2 - 8x - 4 \tag{76}$$

is introduced Then

$$F_1(x) \equiv q_2(x)F_2(x) + F_3(x) \tag{77}$$

The next step in the process is the determination of polynomials $q_3(x)$ and $r_3(x)$, such that $r_3(x)$ is of degree lower than the degree of $F_3(x)$ and that

$$F_2(x) = q_3(x)F_3(x) + r_3(x) \tag{78}$$

Thus, $q_3(x) \equiv -x - 1$, and $r_3(x) \equiv 0$. Therefore (78) becomes in fact

$$(79) \qquad F_2(x) \equiv q_3(x)F_3(x).$$

If (79) is substituted in (77), the identity

$$(80) \qquad F_1(x) \equiv [q_2(x)q_3(x) + 1]F_3(x)$$

is obtained. If (80) and (79) are substituted in (74),

$$(81) \qquad f(x) \equiv \{q_1(x)[q_2(x)q_3(x) + 1] + 2q_3(x)\}F_3(x)$$

is obtained. Now the factor $q_2(x)q_3(x) + 1$ in (80) is a polynomial in x. Hence $F_3(x)$ is a factor of $F_1(x)$. Also, by (81), $F_3(x)$ is a factor of $f(x)$. But $f'(x) \equiv 6F_1(x)$. Hence $F_3(x)$ is a common factor of $f(x)$ and $f'(x)$.

It will now be proved that, if $d(x)$ is any common factor of $f(x)$ and $f'(x)$, then $d(x)$ is a factor of $F_3(x)$. Let there be polynomials $Q(x)$ and $Q_1(x)$ such that

$$(82) \qquad f(x) \equiv Q(x)d(x) \quad \text{and} \quad f'(x) \equiv Q_1(x)d(x).$$

Since $f'(x) \equiv 6F_1(x)$, it follows that

$$(83) \qquad F_1(x) \equiv \tfrac{1}{6}Q_1(x)d(x).$$

Also $(1/6)Q_1(x)$ is a polynomial in x, although its coefficients may not be integers. Hence, by substitution in (74), there is obtained the identity $[Q(x) - q_1(x)\cdot(1/6)Q_1(x)]d(x) \equiv 2F_2(x)$. Hence

$$(84) \qquad F_2(x) \equiv [\tfrac{1}{2}Q(x) - \tfrac{1}{12}q_1(x)Q_1(x)]d(x).$$

Also $(1/2)Q(x) - (1/12)q_1(x)Q_1(x)$ is a polynomial in x. Hence (84) shows that $d(x)$ is a factor of $F_2(x)$. Now, if (83) and (84) are substituted in (77), there is obtained the identity

$$(85) \quad \{\tfrac{1}{6}Q_1(x) - q_2(x)[\tfrac{1}{2}Q(x) - \tfrac{1}{12}q_1(x)Q_1(x)]\}d(x) \equiv F_3(x).$$

The left-hand factor in (85) is a polynomial in x. Hence $d(x)$ is a factor of $F_3(x)$. It has thus been proved that $-F_3(x)$ has the properties (1), (2), (3), which, by definition, characterize the greatest common divisor $g(x)$ of $f(x)$ and $f'(x)$. Hence, if $f(x)$ is the polynomial (70), then $g(x) \equiv x^3 + 5x^2 + 8x + 4$.

By theorem 18, a root s of $g(x) = 0$ of multiplicity m_1 is a root of $f(x) = 0$ of multiplicity $m_1 + 1$. By theorem 17, a root r of

$f(x) = 0$ of multiplicity m is a root of $g(x) = 0$ of multiplicity $m - 1$

Since $g(x)$ is a factor of $f(x)$, there is a polynomial $Q(x)$ such that

$$f(x) \equiv Q(x)g(x) \tag{86}$$

In the preceding illustration $Q(x) \equiv x^3 + x^2 - 4x - 4$ By (86) each root of $Q(x) = 0$ is a root of $f(x) = 0$ It will now be proved that each root of $f(x) = 0$ is a root of $Q(x) = 0$ In fact, it will be proved that, *if r is a root of multiplicity m of $f(x) = 0$, then r is a simple root of $Q(x) = 0$* This will be done by proving first that *$x - r$ is a factor of $Q(x)$ and next that $(x - r)^2$ is not a factor of $Q(x)$*

Now, by the hypothesis that r is a root of multiplicity m of $f(x) = 0$ it follows that there is a polynomial $h(x)$ such that $f(x) = (x - r)^m h(x)$ and $x - r$ is not a factor of $h(x)$ Also, by theorem 17, $(x - r)^{m-1}$ is a common factor of $f(x)$ and $f'(x)$ Hence $(x - r)^{m-1}$ is a factor of $g(x)$ It will now be proved that $(x - r)^m$ is not a factor of $g(x)$ This will be done by showing that if $(x - r)^m$ is a factor of $g(x)$ then there is a contradiction Thus, if $(x - r)^m$ is a factor of $g(x)$, then, by the fact that $g(x)$ is a common factor of $f(x)$ and $f'(x)$ it follows that $(x - r)^m$ is a common factor of $f(x)$ and $f'(x)$ Hence, by theorem 17, $(x - r)^{m+1}$ is a factor of $f(x)$ This contradicts the definition of m Since $(x - r)^{m-1}$ is a factor of $g(x)$ and $(x - r)^m$ is not a factor of $g(x)$, therefore there is a polynomial $h_1(x)$ such that $g(x) \equiv (x - r)^{m-1}h_1(x)$ and $x - r$ does not divide $h_1(x)$

Since $f(x) \equiv (x - r)^m h(x)$ and $g(x) \equiv (x - r)^{m-1}h_1(x)$, the identity (86) becomes

$$(x - r)^m h(x) \equiv Q(x)\ (x - r)^{m-1}h_1(x) \tag{87}$$

Hence $(x - r)h(x) \equiv Q(x)h_1(x)$ Also $x - r$ does not divide $h_1(x)$ Therefore $x - r$ divides $Q(x)$ Next it will be proved that $(x - r)^2$ is not a factor of $Q(x)$ This will be done by showing that, if $(x - r)^2$ is a factor of $Q(x)$, then there is a contradiction Thus, if $(x - r)^2$ is a factor of $Q(x)$, there is a polynomial $Q_1(x)$ such that $Q(x) \equiv (x - r)^2Q_1(x)$ If this result is substituted in (87), and if the result is divided by $(x - r)^m$, the identity $h(x) \equiv (x - r)Q_1(x)h_1(x)$ is obtained Therefore $x - r$ is a factor of $h(x)$ A contradiction of one of the defining properties of $h(x)$ has been obtained This completes the proof of theorem 19

THEOREM 19. *If $f(x) \equiv x^6 + 6x^5 + 9x^4 - 12x^3 - 48x^2 - 48x - 16$ and $g(x) \equiv x^3 + 5x^2 + 8x + 4$, then $g(x)$ is the greatest common divisor of $f(x)$ and its first derivative $f'(x)$. A root of multiplicity m_1 of $g(x) = 0$ is a root of multiplicity $m_1 + 1$ of $f(x) = 0$. If the multiplicity m of a root of $f(x) = 0$ is greater than 1, then this root of $f(x) = 0$ is a root of multiplicity $m - 1$ of $g(x) = 0$. If $Q(x) \equiv x^3 + x^2 - 4x - 4$, then $f(x) \equiv Q(x)g(x)$. If r is a root of multiplicity m of $f(x) = 0$, then r is a simple root of $Q(x) = 0$. If s is a root of $Q(x) = 0$, then s is a root of $f(x) = 0$. Hence $Q(x) = 0$ has no multiple roots.*

One way in which simplification of details in the computation can be achieved will now be illustrated. If

$$f(x) \equiv x^4 - 5x^3 + 6x^2 + 4x - 8, \tag{88}$$

then

$$f'(x) \equiv 4x^3 - 15x^2 + 12x + 4. \tag{89}$$

If $f(x)$ is divided by $f'(x)$, fractions appear as coefficients. But, if $16f(x)$ is divided by $f'(x)$, then all the coefficients will be integers. The tabulation for this division by detached coefficients is:

				4	−5			
4	−15	12	4	16	−80	96	64	−128
				16	−60	48	16	
					−20	48	48	−128
					−20	75	−60	−20
						−27	108	−108

Therefore $16f(x) \equiv (4x - 5)f'(x) - 27x^2 + 108x - 108$. If the notation $c_0 \equiv 16$, $q_1(x) \equiv 4x - 5$, $F_1(x) \equiv f'(x)$, $F_2(x) \equiv -x^2 + 4x - 4$ is introduced, then

$$c_0 f(x) \equiv q_1(x)F_1(x) + 27F_2(x). \tag{90}$$

Again, the tabulation for the division of $F_1(x)$ by $F_2(x)$ by detached coefficients is:

			−4	−1		
−1	4	−4	4	−15	12	4
			4	−16	16	
				1	−4	4
				1	−4	4
					·	

Hence

$$(91) \qquad F_1(x) \equiv (-4x - 1)F_2(x)$$

Therefore by (90) $c_0 f(x) = [(-4x - 1)q_1(x) + 27]F_2(x)$ Therefore $F_2(x)$ is a common divisor of $f(x)$ and $F_1(x)$ Hence if $f(x)$ is defined by (88) then $g(x) = x^2 - 4x + 4$ If the quantity in square brackets is simplified and if the identity is divided by c_0 it is found that

$$(92) \qquad f(x) \equiv Q(x)g(x)$$

with $Q(x) \equiv x^2 - x - 2$ The general argument used in proving theorem 19 shows that the roots of $f(x) = 0$ are the roots of $Q(x) = 0$ and that the roots of $Q(x) = 0$ are simple roots

PROBLEMS

In each problem if $f(x)$ is the polynomial in the equation find $Q(x)$ Solve $Q(x) = 0$ then solve $f(x) = 0$

1 $x^4 - 2x^3 - 11x^2 + 12x + 36 = 0$
2 $x^4 + 7x^3 + 9x^2 - 27x - 54 = 0$
3 $x^4 + 9x^3 - x^2 - 141x - 252 = 0$
4. $x^4 - 14x^3 + 69x^2 - 140x + 100 = 0$
5 $x^4 + x^3 - 7x^2 - x + 6 = 0$
6 $x^4 + 5x^3 + 5x^2 - 5x - 6 = 0$
7 $x^5 - 7x^4 + 19x^3 - 25x^2 + 16x - 4 = 0$
8 $x^5 - 4x^4 + x^3 + 10x^2 - 4x - 8 = 0$
9 $x^5 - 12x^4 + 57x^3 - 134x^2 + 156x - 72 = 0$
10 $x^5 - 7x^4 - 2x^3 + 46x^2 + 65x + 25 = 0$
11 $x^5 - 3x^4 - 5x^3 + 27x^2 - 32x + 12 = 0$
12 $x^5 - 2x^4 - 10x^3 + 8x^2 + 33x + 18 = 0$
13 $x^6 + 8x^5 + 18x^4 - 4x^3 - 47x^2 - 12x + 36 = 0$
14 $x^6 + 3x^5 - 15x^4 - 35x^3 + 90x^2 + 108x - 216 = 0$
15 $x^6 + 2x^5 - 9x^4 - 4x^3 + 31x^2 - 30x + 9 = 0$
16 $x^6 + 4x^5 - 6x^4 - 32x^3 + x^2 + 60x + 36 = 0$

Now let $f(x)$ be the general polynomial (6) of positive degree and let $f'(x)$ be its first derivative It will be proved that $f(x)$ and $f'(x)$ have a greatest common divisor $g(x)$ It will also be proved that if $f(x) = 0$ has a multiple root then $g(x)$ is not a constant that is $g(x)$ actually involves x The converse of this statement will also be proved Then facts which are analogous to those stated in theorem 19 for a particular polynomial will be proved for the general polynomial

If $f(x)$ is linear, then $f'(x)$ is a constant. Hence, if a polynomial in x is a common factor of $f(x)$ and $f'(x)$, then that polynomial is a constant. Therefore, by the definition of greatest common divisor, which was stated preceding (70), $g(x)$ is 1.

If $f(x)$ is not linear, then there are polynomials $q_1(x)$ and $r_1(x)$, such that the degree of $r_1(x)$ is lower than the degree of $f'(x)$, and that

$$(93) \qquad f(x) \equiv q_1(x)f'(x) + r_1(x).$$

If $r_1(x)$ is zero, then $f'(x)$ is a factor of $f(x)$, and $g(x)$ is $(1/na_0)f'(x)$. It will now be proved that, if $r_1(x) \not\equiv 0$, then the greatest common divisor $g(x)$ of $f(x)$ and $f'(x)$ is the greatest common divisor $h_1(x)$ of $f'(x)$ and $r_1(x)$. Thus, it will first be proved that $g(x)$ is a factor of $h_1(x)$. Since $g(x)$ is a factor of $f(x)$ and $f'(x)$, there are polynomials $Q(x)$ and $Q_1(x)$ such that

$$(94) \qquad f(x) \equiv Q(x)g(x), \quad \text{and} \quad f'(x) \equiv Q_1(x)g(x).$$

Hence, by (93),

$$(95) \qquad [Q(x) - q_1(x)Q_1(x)]g(x) \equiv r_1(x).$$

Since $Q(x)$, $q_1(x)$, and $Q_1(x)$ are polynomials, $Q(x) - q_1(x)Q_1(x)$ is a polynomial. Hence (95) shows that $g(x)$ is a factor of $r_1(x)$. Also, by hypothesis, $g(x)$ is a factor of $f'(x)$. Therefore $g(x)$ is a common factor of $f'(x)$ and $r_1(x)$. Hence, by the definition of the greatest common divisor $h_1(x)$ of $f'(x)$ and $r_1(x)$, it is true that $g(x)$ is a factor of $h_1(x)$. It will next be proved that $h_1(x)$ is a factor of $g(x)$. Since $h_1(x)$ is a factor of $f'(x)$ and $r_1(x)$, there are polynomials $S_1(x)$ and $S_2(x)$ such that

$$(96) \qquad f'(x) \equiv S_1(x)h_1(x), \quad \text{and} \quad r_1(x) \equiv S_2(x)h_1(x).$$

Hence, by (93), it is true that

$$(97) \qquad f(x) \equiv [q_1(x)S_1(x) + S_2(x)]h_1(x).$$

Since $q_1(x)$, $S_1(x)$, and $S_2(x)$ are polynomials, $q_1(x)S_1(x) + S_2(x)$ is a polynomial. Hence (97) shows that $h_1(x)$ is a factor of $f(x)$. Now, by hypothesis, $h_1(x)$ is a factor of $f'(x)$. Therefore $h_1(x)$ is a common factor of $f(x)$ and $f'(x)$. Hence, by the definition of the greatest common divisor $g(x)$ of $f(x)$ and $f'(x)$, it follows that $h_1(x)$ is a divisor of $g(x)$. Since $h_1(x)$ is a divisor of $g(x)$ and $g(x)$ is a divisor of $h_1(x)$, they have the same degree. Thus there is a constant k such that $g(x) \equiv k \cdot h_1(x)$. Also $k = 1$, since the lead-

ing coefficient of $g(x)$ is **1** and the leading coefficient of $h_1(x)$ is 1 It has thus been proved that the greatest common divisor $g(x)$ of $f(x)$ and $f'(x)$ is the greatest common divisor $h_1(x)$ of $f'(x)$ and $r_1(x)$

If $r_1(x)$ is of degree 0 then $h_1(x)$ is the constant 1 Hence $g(x) \equiv 1$ If $r_1(x)$ is of degree greater than 0 then there is a polynomial $q_2(x)$ and a polynomial $r_2(x)$ of degree lower than the degree of $r_1(x)$ such that

$$(98) \qquad f'(x) = q_2(x)r_1(x) + r_2(x)$$

If $r_2(x)$ is indeed zero then $r_1(x)$ is a divisor of $f'(x)$ and $h_1(x)$ is the polynomial obtained by dividing $r_1(x)$ by its leading coefficient But if $r_2(x) \neq 0$ then the argument which was applied to (93) to show that $g(x) = h_1(x)$ can be applied to (98) to show that if $h_2(x)$ is the greatest common divisor of $r_1(x)$ and $r_2(x)$ then $h_1(x) = h_2(x)$ and hence $g(x) = h_2(x)$

If this process is repeated the sequence of steps finally terminates in one of two ways Thus it terminates if ever a remainder of zero is obtained If the divisor which yielded this zero remainder is divided by its leading coefficient the result is the greatest common divisor of $f(x)$ and $f'(x)$ If no zero remainder is obtained then the sequence terminates because the degrees of the functions $f(x)$ $f'(x)$ $r_1(x)$ $r_2(x)$ form a sequence of non negative integers such that each integer in the sequence is less than the preceding integer For example the degree of $r_1(x)$ in (93) is at most $n - 2$ and that of $r_2(x)$ in (98) at most $n - 3$ Hence if no zero remainder is obtained then after at most $n - 1$ identities of which (93) and (98) are the first two an identity is obtained in which the degree of the remainder is zero If this last identity is the kth identity then this identity is

$$(99) \qquad r_{k-2}(x) \equiv q_k(x)r_{k-1}(x) + r_k$$

In (99) r_k is indeed a non zero constant Now the argument which was applied to (93) to show that the greatest common divisor of $f(x)$ and $f'(x)$ is the greatest common divisor of $f'(x)$ and $r_1(x)$ can be applied to each of the identities in the sequence Hence finally it is proved that $g(x)$ is the greatest common divisor of $r_{k-1}(x)$ and r_k Since r_k is a non zero constant it follows that $g(x) = 1$ This discussion with theorems 17 and 18 and the proof which follows (86) completes the proof of the following theorem

THEOREM 20. *The general polynomial $f(x)$, of positive degree, and its first derivative $f'(x)$ have a greatest common divisor $g(x)$. This polynomial $g(x)$ is found from one or more identities of the form (93). The equation $f(x) = 0$ has a multiple root if and only if $g(x)$ is not a constant. If the multiplicity m of a root of $f(x) = 0$ is greater than 1, then this root of $f(x) = 0$ is a root of $g(x) = 0$ of multiplicity $m - 1$. If $g(x)$ is not a constant, then a root of $g(x) = 0$ of multiplicity m_1 is a root of $f(x) = 0$ of multiplicity $m_1 + 1$. There is a polynomial $Q(x)$ such that $f(x) \equiv Q(x)g(x)$. The roots of $Q(x) = 0$ are simple roots. They are the distinct roots of $f(x) = 0$.*

It is to be noted especially that in the proof of theorem 20 it was not assumed that the coefficients of $f(x)$ were real numbers. Also it was not assumed that the roots of $f(x) = 0$ were real.

THEOREM 21. *If $f(x)$ is a polynomial with real coefficients, if a and b are real numbers such that $b \neq 0$, and if $a + bi$ is a root of multiplicity m of $f(x) = 0$, then $a - bi$ is a root of multiplicity m of $f(x) = 0$.*

PROOF. Since a_i is a real number, the conjugate $\overline{a_i}$ of the coefficient a_i is a_i. Also, if $c + di$ and $u + vi$ are two complex numbers, the conjugate $\overline{(c + di)(u + vi)}$ of the product $(c + di)(u + vi)$ is the product $(c - di)(u - vi)$ of the conjugates of the two numbers. In particular, $\overline{(c + di)^2} = (c - di)^2$. Repeated use of these facts shows that, if m is any positive integer, and if k is a real number, then $\overline{k(c + di)^m} = k(c - di)^m$. Next, if $c + di$ and $u + vi$ are two complex numbers, then the conjugate $\overline{(c + di) + (u + vi)}$ of their sum is the sum $(c - di) + (u - vi)$ of their conjugates. Hence it follows that the conjugate of the sum of a finite number of complex numbers is the sum of the conjugates of these numbers. If $f(x)$ has the notation (6), then $f(a + bi) = a_0(a + bi)^n + a_1(a + bi)^{n-1} + \cdots + a_{n-1}(a + bi) + a_n$. By the preceding discussion $\overline{f(a + bi)} = \overline{a_0(a + bi)^n} + \overline{a_1(a + bi)^{n-1}} + \cdots + \overline{a_{n-1}(a + bi)} + \overline{a_n} = a_0(a - bi)^n + a_1(a - bi)^{n-1} + \cdots + a_{n-1}(a - bi) + a_n$. Therefore $\overline{f(a + bi)} = f(a - bi)$.

By hypothesis $f(a + bi) = 0$ Therefore $\overline{f(a + bi)} = \bar{0}$ Also $\bar{0} = 0$, and $\overline{f(a + bi)} = f(a - bi)$ Therefore $f(a - bi) = 0$, and $a - bi$ is a root of $f(x) = 0$ Since $b \neq 0$, the numbers $a + bi$ and $a - bi$ are distinct roots of $f(x) = 0$ It has been proved that, if the polynomial $f(x)$ has real coefficients, if a and b are real numbers such that $b \neq 0$, and if $a + bi$ is a root of $f(x) = 0$, then $a - bi$ is a root of $f(x) = 0$ The multiplicity of $a - bi$ for $f(x) = 0$ will be designated by m_1

By theorem 5 there is a polynomial $q_2(x)$ such that

$$(100) \qquad f(x) \equiv [x - (a + bi)][x - (a - bi)]q_2(x)$$

The usual process of finding $q_2(x)$ is first to find $q_1(x)$ such that $f(x) \equiv [x - (a + bi)]q_1(x)$ and next to find $q_2(x)$ such that $q_1(x) \equiv [x - (a - bi)]q_2(x)$ Since $[x - (a + bi)][x - (a - bi)] \equiv x^2 - 2ax + a^2 + b^2$, the polynomial $q_2(x)$ may be found in one step by dividing $f(x)$ by $x^2 - 2ax + a^2 + b^2$ In this process only operations of addition subtraction, and multiplication are performed on the real coefficients of $f(x)$ and of $x^2 - 2ax + a^2 + b^2$ Therefore the coefficients of $q_2(x)$ are real

It will be proved by induction that $m_1 = m$ It is known that $m \geqq 1$ and $m_1 \geqq 1$ It will now be proved that, if $m = 1$ and $m_1 > 1$, then there is a contradiction By (100), if $m_1 > 1$, then $x - (a - bi)$ is a factor of $q_2(x)$ Since the coefficients of $q_2(x)$ are real, the argument preceding (100) is applicable to $q_2(x)$ Therefore there is a polynomial $q_3(x)$ such that $q_2(x) \equiv [x - (a - bi)][x - (a + bi)]q_3(x)$ Hence, by (100), $[x - (a + bi)]^2$ is a factor of $f(x)$ This contradicts the hypothesis that $a + bi$ is a root of multiplicity 1 for $f(x) = 0$ This completes the verification that $m_1 = m$ if $m = 1$

If $m > 1$, then $x - (a + bi)$ is a factor of $q_2(x)$, by (100) By the argument preceding (100), it is known that $x - (a - bi)$ is also a factor of $q_2(x)$ By the hypothesis of the lemma for the induction, if m_2 is the multiplicity of $a + bi$ for $q_2(x) = 0$, then m_2 *is the multiplicity of* $a - bi$ *for* $q_2(x) = 0$ Then there is a polynomial $q_4(x)$ such that $q_2(x) \equiv [x - (a + bi)]^{m_2} [x - (a - bi)]^{m_2} q_4(x)$, and neither $x - (a + bi)$ nor $x - (a - bi)$ is a factor of $q_4(x)$ Substitution in (100) shows that $m = m_2 + 1 = m_1$

Other theorems about polynomial equations are given in the references cited at the end of this book

PROBLEMS

Solve the following equations and thus verify theorem 21 for these equations.

1. $x^3 - 2x^2 - x - 6 = 0.$
2. $x^3 + 5x^2 + 7x + 12 = 0.$
3. $x^5 - x^4 - 7x^3 - 7x^2 + 22x + 24 = 0.$
4. $x^5 - 4x^4 + 2x^3 + 13x^2 - 24x + 12 = 0.$

Show that each of the following equations has at least one multiple root. Solve the equation and thus verify theorem 21.

5. $x^4 - 4x^3 + 10x^2 - 12x + 9 = 0.$
6. $x^4 - 2x^3 + 7x^2 - 6x + 9 = 0.$
7. $x^6 - 3x^5 + 15x^4 - 25x^3 + 60x^2 - 48x + 64 = 0.$
8. $x^6 - 3x^5 + 9x^4 - 13x^3 + 18x^2 - 12x + 8 = 0.$
9. $x^7 - 5x^6 + 14x^5 - 26x^4 + 33x^3 - 29x^2 + 16x - 4 = 0.$
10. $x^7 - x^6 + 4x^5 - 10x^4 + 10x^3 - 16x^2 + 21x - 9 = 0.$
11. $x^5 + 4x^4 + x^3 - 14x^2 - 20x - 8 = 0.$
12. $x^4 + 2x^3 - 2x^2 - 6x + 5 = 0.$
13. $x^4 - 6x^3 + 17x^2 - 24x + 16 = 0.$
14. $x^5 - 2x^4 - 10x^3 + 8x^2 + 33x + 18 = 0.$

CHAPTER 4

ISOLATION AND COMPUTATION OF REAL ROOTS OF REAL POLYNOMIAL EQUATIONS

1. Isolation of real roots by Sturm's theorem illustrated. In this chapter it will be explained how to determine whether a real polynomial equation has any real roots and how to compute in decimal form each real root which it may have.

Sturm's theorem concerns a real polynomial equation which has no multiple roots. Hence in the following numerical illustration of the use of Sturm's theorem *the first step is to determine the greatest common divisor $g(x)$ of $f(x)$ and $f'(x)$*. If $f(x)$ designates the polynomial which forms the left-hand side of the equation

$$x^4 - 4x^3 + 4x^2 + 4x - 3 = 0 \tag{1}$$

then

$$f'(x) = 4x^3 - 12x^2 + 8x + 4 \tag{2}$$

If $F_1(x) = x^3 - 3x^2 + 2x + 1$, then $f'(x) = 4F_1(x)$. Now if $f(x)$ is divided by $F_1(x)$, the equation $f(x) = (x - 1)F_1(x) - x^2 + 5x - 2$ results. Hence if $F_2(x) = x^2 - 5x + 2$, then $4f(x) = (x - 1)f'(x) - 4F_2(x)$. If the notations

$$c_0 = 4, \quad q_1(x) = x - 1, \quad r_1(x) = 4(-x^2 + 5x - 2) \tag{3}$$

are used, this identity becomes

$$c_0 f(x) = q_1(x)f'(x) + r_1(x) \tag{4}$$

This identity may be roughly checked. Thus if x is replaced by 2, then the equation $c_0 f(2) = q_1(2)f'(2) + r_1(2)$ results. By (3) and (2) this equation is $4 \cdot 5 = 1 \cdot 4 + 4 \cdot 4$. Since this last equation is a true relation between numbers and since 2 was a value of x chosen at random, it is probable that the identity (4) is correct.

The identity (4) is the first identity in the usual process of finding the greatest common divisor $g(x)$ of $f(x)$ and $f'(x)$. In order

that (4) and subsequent identities may be given a unified notation, $f_0(x)$, $f_1(x)$, and $f_2(x)$ are defined by

(5) $$f_0(x) \equiv f(x), \quad f_1(x) \equiv f'(x), \quad f_2(x) \equiv -r_1(x).$$

Then (4) becomes

(6) $$c_0 f_0(x) \equiv q_1(x) f_1(x) - f_2(x).$$

It is to be noted especially that $f_2(x)$ is the negative of the remainder $r_1(x)$, which was obtained in (4).

Next, a positive constant c_1, and polynomials $q_2(x)$ and $f_3(x)$, will be found, such that

(7) $$c_1 f_1(x) \equiv q_2(x) f_2(x) - f_3(x)$$

and the degree of $f_3(x)$ is less than the degree of $f_2(x)$. Since $f_1(x) \equiv 4F_1(x)$ and $f_2(x) \equiv 4F_2(x)$, therefore the computation is more simple if $F_1(x)$ is divided by $F_2(x)$. The identity $F_1(x) \equiv (x + 2)F_2(x) + 10x - 3$ results. Hence $4F_1(x) \equiv (x + 2)4F_2(x) + 4(10x - 3)$. Therefore (7) is true with

(8) $$c_1 = 1, \quad q_2(x) \equiv x + 2, \quad f_3(x) \equiv 4(-10x + 3).$$

Finally, a positive constant c_2, and polynomials $q_3(x)$ and $f_4(x)$, will be found such that

(9) $$c_2 f_2(x) \equiv q_3(x) f_3(x) - f_4(x)$$

and the degree of $f_4(x)$ is less than the degree of $f_3(x)$. Since $f_3(x)$ is a linear function, it follows that $f_4(x)$ will be a constant. Since $f_2(x) \equiv 4F_2(x)$ and $f_3(x) \equiv 4(-10x + 3)$, therefore the computation is performed with $F_2(x)$ and $-10x + 3$. Since division of $F_2(x)$ by $-10x + 3$ would introduce fractional coefficients, $100F_2(x)$ is divided by $-10x + 3$. The result is $100F_2(x) \equiv (-10x + 47)(-10x + 3) + 59$. If this result is multiplied by 4, and if the relations $f_2(x) \equiv 4F_2(x)$ and $f_3(x) \equiv 4(-10x + 3)$ are used, it is found that $100f_2(x) \equiv (-10x + 47)f_3(x) + 236$. Therefore (9) is true with

(10) $$c_2 = 100, \quad q_3(x) \equiv -10x + 47, \quad f_4(x) \equiv -236.$$

The identities (6), (7), and (9) will now be used to determine the greatest common divisor of $f(x)$ and $f'(x)$. The argument is precisely that which was applied from (93) to (99) in chapter 3. The greatest common divisor of $f_0(x)$ and $f_1(x)$ is, by (6), the

greatest common divisor of $f_1(x)$ and $f_2(x)$, and hence, by (7), the greatest common divisor of $f_2(x)$ and $f_3(x)$, and hence, by (9), of $f_3(x)$ and $f_4(x)$. Since $f_4(x)$ is a non zero constant, it follows that the greatest common divisor of $f(x)$ and $f'(x)$ is 1. By theorem 20 of chapter 3 equation (1) has no multiple roots.

The Sturm functions for equation (1) are the functions $f_0(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_4(x)$. Therefore

$$
\begin{aligned}
f_0(x) &\equiv x^4 - 4x^3 + 4x^2 + 4x - 3, \\
f_1(x) &\equiv 4(x^3 - 3x^2 + 2x + 1), \\
f_2(x) &\equiv 4(x^2 - 5x + 2), \qquad (11) \\
f_3(x) &\equiv 4(-10x + 3) \\
f_4(x) &= -236
\end{aligned}
$$

These functions are also designated by f_0, f_1, f_2, f_3, f_4. They will be used later to obtain information about whatever real roots (1) may have.

PROBLEMS

In each of the following problems show that the equation has no multiple root. Write all the identities used in this process and roughly check each identity. Write the Sturm functions.

1. $x^3 - 3x^2 - 15x + 1 = 0$
2. $x^3 + 3x^2 + 12x - 10 = 0$
3. $x^3 + 3x^2 + 6x - 12 = 0$
4. $x^3 - 3x^2 - 6x + 1 = 0$
5. $x^3 + 12x^2 + 12x - 11 = 0$
6. $x^3 - 15x^2 + 6x - 7 = 0$
7. $x^4 + 4x^3 - 4x^2 + 4x - 1 = 0$
8. $x^4 - 4x^3 + 4x^2 - 6x + 1 = 0$
9. $x^3 - 6x^2 + 2 = 0$
10. $x^3 + 3x^2 - 7 = 0$
11. $x^3 - 3x + 5 = 0$
12. $x^3 - 6x + 2 = 0$
13. $x^4 + 8x^3 - 4x + 1 = 0$
14. $x^4 - 8x^2 + 8x - 2 = 0$
15. $x^4 + 2x^3 + 4x^2 + 2x + 2 = 0$
16. $x^4 - 2x^3 - x^2 + 2x - 1 = 0$

The second step in the use of Sturm's theorem is the tabulation of signs. The left-hand column of the table lists the Sturm functions in order and a symbol V which will be explained later. By theorem 11 of chapter 3 an upper bound for the real roots of (1) is 5 and a lower bound is -3. Hence the top row of the table lists in order values of x between -3 and 5. The symbols P and $-P$ in this row will be explained later.

If $x = 0$ the values of the Sturm functions (11) are -3, 4, 8, 12, -236 respectively. The signs of these numbers are $- + + + -$. In the table these signs are listed in the column with the

value 0 of x at the top. Again, if $x = 1$, the signs are $+ + - - -$. These signs are listed in the column with the value 1 of x at the top. All the signs in the table except those in the column headed by P and those in the column headed by $-P$ are found in this way. If P is a positive number, the signs of the leading terms of $f_0(P)$, $f_1(P)$, $f_2(P)$, $f_3(P)$, f_4 are $+ + + - -$. These signs are listed in the column headed by P. The signs of the leading terms of $f_0(-P)$, $f_1(-P)$, $f_2(-P)$, $f_3(-P)$, f_4 are $+ - + + -$. These signs are listed in the column headed by $-P$.

	$-P$	$\cdots$	-3	-2	-1	0	1	2	3	4	5	$\cdots$	P
f_0	+	$\cdots$	+	+	+	−	+	+	+	+	+	$\cdots$	+
f_1	−	$\cdots$	−	−	−	+	+	+	+	+	+	$\cdots$	+
f_2	+	$\cdots$	+	+	+	+	−	−	−	−	+	$\cdots$	+
f_3	+	$\cdots$	+	+	+	+	−	−	−	−	−	$\cdots$	−
f_4	−	$\cdots$	−	−	−	−	−	−	−	−	−	$\cdots$	−
V	3	$\cdots$	3	3	3	2	1	1	1	1	1	$\cdots$	1

The last row of the table will now be explained. The sequence of signs in the column headed by 0 is $- + + + -$. The first two signs $- +$ in this sequence present a variation in sign because they are opposite. The second and third signs $+ +$ do not present a variation in sign. The third and fourth signs $+ +$ do not present a variation. The last two signs $+ -$ present a variation in sign. Therefore the number of variations in this sequence is 2. This fact is recorded by the entry 2 at the bottom of this column. The number of variations in this sequence is designated by V_0. Therefore $V_0 = 2$. Again, the sequence of signs in the column with -1 at the top is $+ - + + -$. Since this sequence presents three variations, the entry 3 appears in the last row in this column. The number of variations in this sequence is designated by V_{-1}. Therefore $V_{-1} = 3$. Each entry in the last row of the table is obtained in this manner. V_c designates the number of variations in the sequence of signs in the column with c at the top.

Sturm's theorem states that, if a and b are real numbers, neither of which is a root of (1), and if $a < b$, then $V_a \geqq V_b$ and the number of real roots of (1) between a and b is $V_a - V_b$. For example, since $V_{-2} - V_{-1} = 0$, there is no real root of (1) between -2 and -1. Since $V_{-1} - V_0 = 1$, there is one real root

between -1 and 0 Since $V_0 - V_1 = 1$, there is one real root between 0 and 1

In the works to which reference is made at the end of this book it is proved that, if $g(x)$ is the real polynomial $b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$, then there is a positive number k, depending on the coefficients in $g(x)$ such that, if $P > k$, then the sign of $g(P)$ is the same as the sign of b_0P^n It follows that there is a positive number c, depending on the coefficients in $g(x)$, such that, if $Q < -c$, then the sign of $g(Q)$ is the same as the sign of b_0Q^n Now let $g(x)$ be taken in turn to be the Sturm functions (11) Then there is a positive number P large enough that simultaneously the signs of $f_0(P)$ $f_1(P)$ $f_2(P)$, $f_3(P)$ f_4 are the signs in the column headed by P, and the signs of $f_0(-P)$, $f_1(-P)$, $f_2(-P)$, $f_3(-P)$, f_4 are the signs in the column headed by $-P$ Since $V_1 - V_P = 0$, there is no real root greater than 1 Since $V_{-P} - V_{-1} = 0$, there is no real root less than -1

PROBLEMS

Tabulate the signs of the Sturm functions of the equations in the problems of the preceding set Isolate the real roots For each real root determine consecutive integers such that the root is between these integers

2 Sturm's theorem In section 1 Sturm's theorem was used to isolate the real roots of the numerical equation (1) This theorem will now be proved It concerns the general real polynomial equation $f(x) = 0$, which has no multiple roots, and arbitrary real numbers a and b neither of which is a root of $f(x) = 0$ It is assumed that $a < b$

In the following proof there are four parts In (i) the Sturm functions for $f(x)$ will be defined by a sequence of identities, and the symbol V_c will be defined for the arbitrary real number c In (ii) the closed interval of real numbers from a to b will be separated into appropriate subintervals and all the possible types of subintervals will be determined In (iii) the value of $V_c - V_d$ will be determined if the closed interval from c to d is in turn a subinterval of each of the types in (ii) In (iv) the value of $V_a - V_b$ will be determined

(i) By hypothesis n is a positive integer, the coefficients a_0, a_1, $\cdots$, a_n in

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \tag{12}$$

are real numbers, and $a_0 \neq 0$. Also $f(x) = 0$ has no multiple roots The first derivative of $f(x)$ is designated by $f'(x)$. By theorem 20 of chapter 3 the greatest common divisor of $f(x)$ and $f'(x)$ is the constant 1. If $f(x)$ and $f'(x)$ are designated by $f_0(x)$ and $f_1(x)$, then there are polynomials $q_1(x)$ and $f_2(x)$, and a positive constant c_0, such that

$$c_0 f_0(x) \equiv q_1(x) f_1(x) - f_2(x) \tag{13}$$

and the degree of $f_2(x)$ is less than the degree of $f_1(x)$. If $f_2(x)$ is a constant, the Sturm functions are $f_0(x)$, $f_1(x)$, and $f_2(x)$. If $f_2(x)$ is not a constant, then there is a sequence (14) of identities such that in the last identity $f_l(x)$ is a non-zero constant:

$$\begin{aligned}
c_0 f_0(x) &\equiv q_1(x) f_1(x) - f_2(x), \\
c_1 f_1(x) &\equiv q_2(x) f_2(x) - f_3(x), \\
&\ \ \vdots \\
c_{k-2} f_{k-2}(x) &\equiv q_{l-1}(x) f_{l-1}(x) - f_l(x).
\end{aligned} \tag{14}$$

It is to be noted especially that $c_0, c_1, \cdots, c_{k-2}$ are positive constants, that $q_1(x), \cdots, q_{k-1}(x), f_0(x), \cdots, f_{l-1}(x)$ are polynomials in x, and that the degree of $f_i(x)$ is lower than the degree of $f_{i-1}(x)$. In fact, $f_2(x), \cdots, f_l(x)$ are the negatives of the remainders in the identities obtained in the usual process of finding the greatest common divisor of $f(x)$ and $f'(x)$. *The Sturm functions for $f(x)$ are the functions $f_0(x), f_1(x), \cdots, f_l(x)$.* They are also designated by $f_0, f_1, \cdots, f_l$.

Now let c be any real number which is not a root of $f(x) = 0$. In the sequence $f_0(c), f_1(c), \cdots, f_k(c)$ of numbers it is known that $f_0(c) \neq 0$ and that $f_l(c) \neq 0$. It may be that no number in this sequence is zero, but it may be that one or more of the numbers $f_1(c), \cdots, f_{k-1}(c)$ are zero. Let a new list be formed by deleting each of these numbers which is zero. Then each number in this final list is not zero and hence has a sign. This sequence of signs may present one or more variations in sign, or it may present no variation in sign. *The symbol V_c designates the number of variations in sign presented by the list $f_0(c), f_1(c), \cdots, f_k(c)$ after zero terms are discarded.*

(ii) Since $f(x)$ is of degree n, the equation $f(x) = 0$ has at most n real roots An analogous statement holds for each of the equations $f_1(x) = 0, \quad , f_{k-1}(x) = 0$ Hence all the real roots of all the equations $f_0(x) = 0, \quad , f_{k-1}(x) = 0$ constitute a finite set If c and d are arbitrary real numbers such that $c < d$, then $[c, d]$ *designates the closed interval from c to d*, that is, all numbers u such that $c \leq u \leq d$ Now the interval $[c, d]$ and all the roots of all of the equations $f_0(x) = 0, \quad , f_{k-1}(x) = 0$ may have one of the following relations The interval will be said to be of *type I* if no one of these roots is in the interval It is of *type II* if c is a root of at least one of these equations and if no other number in $[c, d]$ is a root of any of these equations It is of *type III* if d is a root of at least one of these equations and if no other number in $[c, d]$ is a root of any of these equations It is of *type IV* if there is a number s such that $c < s < d$ $f_0(s) \neq 0$, and s is a root of at least one of $f_1(x) = 0, \quad , f_{k-1}(x) = 0$, and if s is the only number in $[c, d]$ which is a root of any of the equations $f_0(x) = 0, \quad , f_{k-1}(x) = 0$ It is *of type V* if there is a number s such that $c < s < d$ and $f_0(s) = 0$ and if s is the only number in $[c\ d]$ which is a root of any of the equations $f_0(x) = 0, \quad , f_{k-1}(x) = 0$ There are other possible types of relation which an arbitrary interval $[c, d]$ and all the roots of all the equations $f_0(x) = 0, \quad , f_{k-1}(x) = 0$ may have However these five types are the only types which appear in the following proof

It will now be explained how $[a\ b]$ is separated into a finite number of closed subintervals such that each subinterval is of one of the preceding five types and two subintervals have either no point, or one end point and no other point, in common Thus, it may be that $[a\ b]$ is of type I, or of type II, or of type III In each of these cases there is only one subinterval, that interval being $[a, b]$ itself Again, it may be that a is a root of one of $f_0(x) = 0, \quad , f_{k-1}(x) = 0$, that b is also a root of one of these equations, and that no other number in $[a, b]$ is a root of any of these equations Then $[a, b]$ is separated into two subintervals by an arbitrary point u such that $a < u < b$ Then $[a, u]$ is of type II, and $[u, b]$ is of type III Otherwise, there is at least one of the roots of all of the equations $f_0(x) = 0, \quad , f_{k-1}(x) = 0$ which is between a and b, not equal to a, and not equal to b If there is exactly one such root r in $[a, b]$, then there are two numbers b_1 and b_2 such that $a < b_1 < r < b_2 < b$ Also $[a, b_1]$ is of type I or II, $[b_1, b_2]$ of type IV or V, and $[b_2, b]$ of type I or III

If there are several such roots in $[a, b]$, then the number m of these roots is finite, and the notation $r_1, \cdots, r_m$ for these roots can be chosen so that $a < r_1 < \cdots < r_m < b$. Then there are numbers $b_1, \cdots, b_{m+1}$ such that $a < b_1 < r_1 < \cdots < b_m < r_m < b_{m+1} < b$. Then $[a, b]$ is separated into the intervals $[a, b_1]$, $[b_1, b_2], \cdots, [b_m, b_{m+1}]$, $[b_{m+1}, b]$. Also $[a, b_1]$ is of type I or II, $[b_{m+1}, b]$ is of type I or III, and each of the other intervals is of type IV or V.

(iii) One property of continuous functions will be used in the following proof. This is the property that, if $g(x)$ is a continuous function, and if the curve whose equation is $y = g(x)$ is on one side of the X-axis if $x = c$ and on the other side if $x = d$, then somewhere between c and d the curve crosses the X-axis. This property is also expressed by the statement that, if $g(c) > 0$, and if $g(x) \neq 0$ for each x in $[c, d]$, then $g(d) > 0$. It is true that a polynomial in x is a continuous function of x and that $f_0(x), \cdots, f_{l-1}(x)$ are polynomials in x. Therefore this property is a property of each of $f_0(x), \cdots, f_{l-1}(x)$.

It will now be proved that, *if* $[c, d]$ *is of type I, then* $V_c - V_d = 0$. Thus, by the property mentioned, the signs of $f_0(c)$ and $f_0(d)$ are $+\ +$, or they are $-\ -$. Also, by this property, the signs of $f_1(c)$ and $f_1(d)$ are $+\ +$, or they are $-\ -$. Thus the entries in the first two rows of the columns headed by c and d form one of the following four tables:

$$\begin{matrix} + & + \\ + & + \end{matrix}, \quad \begin{matrix} + & + \\ - & - \end{matrix}, \quad \begin{matrix} - & - \\ + & + \end{matrix}, \quad \begin{matrix} - & - \\ - & - \end{matrix}. \tag{15}$$

In the first and last tables these two rows contribute no variation to V_c and no variation to V_d. In the second and third tables these two rows contribute one variation to V_c and one variation to V_d. Hence the number of variations which the first two rows contribute to V_c equals the number of variations which they contribute to V_d. This same argument is applicable to the rows for f_1 and f_2, and to each set of two adjacent rows. Therefore $V_c - V_d = 0$.

It will now be proved that, *if* $[c, d]$ *is of type IV, then* $V_c - V_d = 0$. This will be done by inserting the column headed by s between the column headed by c and the column headed by d. By the hypothesis $f(s) \neq 0$ in the definition of type IV, and by the property of continuous functions which was mentioned above, the

signs of $f_0(c)$ $f_0(s)$ $f_0(d)$ are $+ + +$ or $- - -$ If $f_1(s) \neq 0$ then the entries in the second row of these three columns are $+ + +$ or $- - -$ Therefore these two rows of these three columns form one of the four tables

$$(16)\qquad \begin{array}{ccc} + & + & + \\ + & + & + \end{array} \qquad \begin{array}{ccc} + & + & + \\ - & - & - \end{array} \qquad \begin{array}{ccc} - & - & - \\ + & + & + \end{array} \qquad \begin{array}{ccc} - & - & - \\ - & - & - \end{array}$$

Hence if $f_1(s) \neq 0$ the number of variations which the first *two* rows contribute to V_c equals the number which they contribute to V_d

If $f_1(s) = 0$ it will be proved that the number of variations which the first *three* rows contribute to V_c equals the number which they contribute to V_d If the entries for $f_1(c)$ and $f_1(d)$ are omitted the entries for the first three rows of these columns form one of the following tables

$$(17)\qquad \begin{array}{ccc} + & + & + \\ & 0 & \\ - & - & - \end{array} \qquad \begin{array}{ccc} - & - & - \\ & 0 & \\ + & + & + \end{array}$$

Thus by the property of continuous functions the signs of $f_0(c)$ $f_0(s)$ $f_0(d)$ are $+ + +$ or $- - -$ Also by the first equation in (14) it is true that $c_0 f_0(s) = q_1(s) f_1(s) - f_2(s)$ Since $f_1(s) = 0$ and $c_0 f_0(s) \neq 0$ it follows that $f_2(s) \neq 0$ Since $c_0 > 0$ the sign of $f_2(s)$ is opposite to the sign of $f_0(s)$ Hence the column headed by s is the middle column in one of the tables in (17) Finally by the property of continuous functions the entries in the third row are $+ + +$ or $- - -$ This completes the proof of the statement about the tables (17) Now $f_1(c)$ may have the entry $+$ or the entry $-$ and so may $f_1(d)$ Thus the first table in (17) is completed in one of the following ways

$$(18)\qquad \begin{array}{ccc} + & + & + \\ + & 0 & + \\ - & - & - \end{array} \qquad \begin{array}{ccc} + & + & + \\ + & 0 & - \\ - & - & - \end{array} \qquad \begin{array}{ccc} + & + & + \\ - & 0 & + \\ - & - & - \end{array} \qquad \begin{array}{ccc} + & + & + \\ - & 0 & - \\ - & - & - \end{array}$$

In each of the tables in (18) there is one variation contributed to V_c and one variation to V_d Again the second table in (17) is completed similarly in one of four ways and always there is one variation contributed to V_c and one variation to V_d This completes the proof that if $f_1(s) = 0$ then the number of variations

contributed to V_c by the rows for f_0, f_1, f_2 equals the number of variations contributed to V_d by these three rows.

The remaining rows in these three columns can be treated in one of these two ways. If these columns commenced as in one of the tables in (16), and if $f_2(s) \neq 0$, then the second and third rows would yield one of the tables in (16). If these columns commenced as in one of the tables in (16), and if $f_2(s) = 0$, then the second, third, and fourth rows would yield one of the tables obtained from (17). In general, the rows of the three columns are considered in sets of two rows each, or in sets of three rows each. The number of variations contributed to V_c by such a set of rows equals the number contributed to V_d by this set. This completes the proof that, if $[c, d]$ is of type IV, then $V_c - V_d = 0$.

It will now be proved that, *if* $[c, d]$ *is of type II, then* $V_c - V_d = 0$. By the definition of type II the columns headed by c and d are related to each other as, in the proof for type IV, the columns which are headed by s and d there are related to each other. The proof for type IV also yields the fact that there $V_s - V_d = 0$. Therefore here $V_c - V_d = 0$.

It will be proved next that, *if* $[c, d]$ *is of type III, then* $V_c - V_d = 0$. By the definition of type III, the columns headed by c and d are related to each other as, in the proof for type IV, the columns which are headed by c and s there are related to each other. The proof for type IV also yields the fact that there $V_c - V_s = 0$. Therefore here $V_c - V_d = 0$.

It will now be proved that, *if* $[c, d]$ *is of type V, then* $V_c - V_d = 1$. This will be done by inserting the column headed by s between the column headed by c and the column headed by d. By hypothesis $f_0(s) = 0$. It will be proved first that $f_1(s) \neq 0$, by showing that, if $f_1(s)$ is zero, then there is a contradiction. If $f_0(s) = 0$ and $f_1(s) = 0$, then, by theorem 18 of chapter 3, s is a multiple root of $f(x) = 0$. This contradicts the hypothesis that $f(x) = 0$ has no multiple roots. Now, by the fact that $f_1(s) \neq 0$ and by the property of continuous functions, the signs of $f_1(c)$, $f_1(s)$, $f_1(d)$ are $+ + +$ or $- - -$. Therefore, if the entries for $f_0(c)$ and $f_0(d)$ are omitted, the entries for the first two rows of these three columns form one of the tables:

$$(19) \qquad \begin{matrix} & 0 & \\ + & + & + \end{matrix}\,, \qquad \begin{matrix} & 0 & \\ - & - & - \end{matrix}\,.$$

Now $f_1(x)$ is the first derivative of $f(x)$ Therefore, for the first table in (19) $f(x)$ is a function whose first derivative is positive in the interval $[c, d]$ Hence $f(x)$ is an increasing function in that interval Therefore the entry for $f_0(c)$ is $-$, and the entry for $f_0(d)$ is $+$, and this table becomes

$$(20)\qquad \begin{matrix} - & 0 & + \\ + & + & + \end{matrix}$$

Again, for the second table in (19), $f(x)$ is a function whose first derivative is negative in $[c, d]$ Hence $f(x)$ is a decreasing function in $[c, d]$ Therefore this table becomes

$$(21)\qquad \begin{matrix} + & 0 & - \\ - & - & - \end{matrix}$$

Therefore f_0 and f_1 contribute one variation to V_c and no variation to V_d

If $f_2(s) \neq 0$, then the rows for f_1 and f_2 yield one of the tables in (16) If $f_2(s) = 0$, then the rows for f_1, f_2, and f_3 yield one of the tables obtained from (17) This process is repeated until all the rows for $f_1, \quad , f_k$ in these columns have been considered Therefore the number of variations contributed to V_c by the rows for $f_1, \quad , f_k$ equals the number of variations contributed to V_d by the rows for $f_1, \quad , f_k$

Therefore the number of variations contributed to V_c by the rows for f_0 $f_1, \quad , f_k$ is one more than the number of variations contributed to V_d by these rows Therefore $V_c - V_d = 1$, if $[c, d]$ is of type V

(iv) It will now be proved that $V_a - V_b$ *equals the number of real roots in* $[a\ b]$ In (ii) it was explained how to separate $[a, b]$ into subintervals It may be that there is only one subinterval Then, as explained in (ii), this interval is $[a, b]$ itself, and it is of type I, II, or III Therefore, by (iii), $V_a - V_b = 0$ Also, by the definitions of the types I, II, and III, then there is no real root of $f(x) = 0$ in $[a, b]$ Therefore, in this case, $V_a - V_b$ equals the number of real roots in $[a\ b]$

Again, it may be that there are two subintervals Then, as explained in (ii), there is a real number u such that $a < u < b$ and $[a, u]$ is of type II and $[u, b]$ is of type III Also, $V_a - V_b =$

$(V_a - V_u) + (V_u - V_b)$. By (iii) $V_a - V_u = 0$, and $V_u - V_b = 0$. Therefore $V_a - V_b = 0$. Also, by the definition of the types II and III, there is no real root of $f(x) = 0$ in $[a, u]$, and there is no real root of $f(x) = 0$ in $[u, b]$. Therefore there is no real root of $f(x) = 0$ in $[a, b]$. Therefore, in this case, $V_a - V_b$ equals the number of real roots in $[a, b]$.

Otherwise, as explained in (ii), there is a positive integer m, and there are real numbers $b_1, \cdots, b_{m+1}$, such that $[a, b]$ is separated into $[a, b_1]$, $[b_1, b_2]$, $\cdots$, $[b_m, b_{m+1}]$, $[b_{m+1}, b]$. Then $V_a - V_b = (V_a - V_{b_1}) + (V_{b_1} - V_{b_2}) + \cdots + (V_{b_m} - V_{b_{m+1}}) + (V_{b_{m+1}} - V_b)$. Now a difference, in parentheses on the right-hand side of this equation, equals one if there is a root of $f(x) = 0$ in the corresponding interval, but the difference equals zero if there is no root of $f(x) = 0$ in this interval. Therefore, in this case, $V_a - V_b$ equals the number of real roots of $f(x) = 0$ in $[a, b]$.

Sturm's theorem. *Let $f(x)$ be a real polynomial in x of positive degree. Let $f(x) = 0$ have no multiple roots. Let a and b be real numbers such that $a < b$, $f(a) \neq 0$, and $f(b) \neq 0$. Let the Sturm functions for $f(x)$ be defined as in (14). Then the exact number of real roots of $f(x) = 0$ which are between a and b is $V_a - V_b$.*

In the references cited at the end of this book there are methods of avoiding computation in the use of Sturm's theorem. There is also a modified Sturm's theorem which is applicable even if the equation has multiple roots.

PROBLEMS

In each of the following problems show that the equation has no multiple root, tabulate the signs of the Sturm functions, and isolate the real roots. Determine consecutive integers between which each real root lies.

1. $x^3 - 7x^2 + 18x - 13 = 0$.
2. $x^3 - 10x^2 + 33x - 31 = 0$.
3. $x^3 - 5x^2 + 7x - 1 = 0$.
4. $x^4 - x^2 - x - 6 = 0$.
5. $x^4 - 4x^3 + 7x^2 - 9x + 3 = 0$.
6. $x^4 - 3x^3 + 3x^2 - 4x + 2 = 0$.
7. $x^5 - 5x^3 + 5x^2 - 5x + 3 = 0$.
8. $x^5 - 10x^3 + 15x^2 - 8 = 0$.
9. $x^4 - 2x^3 + x^2 - 2x - 5 = 0$.
10. $x^4 - 3x^3 + 2x^2 - x - 1 = 0$.

3. Descartes' rule of signs. Several illustrations of the use of Descartes' rule of signs will be given in this section, but the rule will not be proved in this book. As given in the references, the proof is long but not difficult.

DESCARTES RULE OF SIGNS *If $f(x)$ is a real polynomial in x of positive degree then the number of positive roots of the equation $f(x) = 0$ is either equal to the number of variations in sign presented by the coefficients of $f(x)$ or less than this number by an even integer*

In the equation

$$x^4 - 4x^3 + 4x^2 + 4x - 3 = 0 \tag{22}$$

the signs of the coefficients are $+ - + + -$ These signs present three variations By Descartes' rule the number of positive roots is three or one By Sturm s theorem it was proved in section 1 that there is one positive root of this equation

In the equation

$$x^6 - 4x^5 + 15x^4 - 28x^3 + 49x^2 - 42x + 36 = 0 \tag{23}$$

the signs of the coefficients are $+ - + - + - +$ These signs present six variations By Descartes rule the number of positive roots of (23) is six four two or zero Sturm s theorem would show that there are no real roots of (23)

In the equation

$$x^3 - 3x^2 + 5x - 6 = 0 \tag{24}$$

the signs of the coefficients are $+ - + -$ These signs present three variations By Descartes rule the number of positive roots of (24) is three or one Sturm s theorem would show that there is one positive root of (24)

In the equation

$$x^3 - x^2 - 2x - 2 = 0 \tag{25}$$

the signs of the coefficients are $+ - - -$ These signs present one variation By Descartes rule there is one positive root of (25) In this case Descartes rule gives the exact number of positive roots

In the equation

$$x^3 - 1 = 0 \tag{26}$$

the signs are $+ -$ These signs present one variation Therefore by Descartes rule there is one positive root This fact was used in chapter 1

If x is replaced by $-y$ in the equation

$$x^3 + 2x^2 - x + 5 = 0, \tag{27}$$

there results the equation

$$-y^3 + 2y^2 + y + 5 = 0. \tag{28}$$

By Descartes' rule there is one positive root of (28). Therefore there is one negative root of (27).

4. Horner's method. By Descartes' rule of signs the equation

$$x^3 - 7x^2 + 14x - 7 = 0 \tag{29}$$

has at least one positive root. If $f(x) \equiv x^3 - 7x^2 + 14x - 7$, then $f(0) = -7, f(1) = 1, f(2) = 1, f(3) = -1, f(4) = 1$. Therefore the curve whose equation is $y = f(x)$ crosses the X-axis between 0 and 1, between 2 and 3, and between 3 and 4. Therefore (29) has one root in the interval [0, 1], one root in the interval [2, 3], and one root in the interval [3, 4].

Horner's method of calculating the root of (29) which lies between 2 and 3 will now be explained. This root x will be known if a number u can be found such that

$$x = 2 + u. \tag{30}$$

Since x is a root of (29), u is a root of the equation which is obtained from (29) by the transformation (30). This equation in u could be found by substitution from (30) in (29). Thus, if the operations indicated in $(2 + u)^3 - 7(2 + u)^2 + 14(2 + u) - 7$ are performed, and if like powers of u are combined, the equation

$$u^3 - u^2 - 2u + 1 = 0 \tag{31}$$

is obtained. The polynomial in u which constitutes the left-hand side of (31) will be designated by $U(u)$. Therefore

$$f(x) \equiv U(u) \tag{32}$$

under the transformation (30).

A more simple method of obtaining (31) will now be explained. Before the coefficients in $U(u)$ have been found, they will be designated by c_0, c_1, c_2, c_3 respectively. Hence $U(u) \equiv c_0u^3 + c_1u^2$

$+ c_2u + c_3$ If u in (32) is replaced by $x - 2$ then (29) is obtained This fact is stated by the identity

$$(33)\quad c_0(x-2)^3 + c_1(x-2)^2 + c_2(x-2) + c_3 \equiv x^3 - 7x^2 + 14x - 7$$

Therefore

$$(34)\quad [c_0(x-2)^2 + c_1(x-2) + c_2](x-2) + c_3 \equiv x^3 - 7x^2 + 14x - 7$$

This identity shows that if $f(x)$ is divided by $x - 2$ until a constant remainder is obtained then this remainder is the required value of c_3 If this division is accomplished synthetically it is exhibited in the table

$$(35)\quad \begin{array}{rrrr|l} 1 & -7 & 14 & -7 & 2 \\ & 2 & -10 & 8 & \\ \hline 1 & -5 & 4 & 1 & \end{array}$$

The entry 1 in the third row and fourth column is c_3

The quotient in (34) is the quotient indicated by (35) Hence

$$(36)\quad c_0(x-2)^2 + c_1(x-2) + c_2 \equiv x^2 - 5x + 4$$

Therefore

$$(37)\quad [c_0(x-2) + c_1](x-2) + c_2 \equiv x^2 - 5x + 4$$

This identity shows that if the quotient $x^2 - 5x + 4$ in the first step (35) is itself divided by $x - 2$ until a constant remainder is obtained then this remainder is the required value of c_2 This division is also accomplished synthetically A table should be constructed in the usual manner to exhibit this synthetic substitution This table and (35) may be combined in the table

$$(38)\quad \begin{array}{rrr|r|l} 1 & -7 & 14 & -7 & 2 \\ & 2 & -10 & 8 & \\ \hline 1 & -5 & 4 & 1 & \\ & 2 & -6 & & \\ \hline 1 & -3 & -2 & & \end{array}$$

The entry -2 in the fifth row and third column of (38) is c_2 The quotient in (37) is the quotient indicated in the last line of (38)

Hence

$$c_0(x - 2) + c_1 \equiv x - 3. \tag{39}$$

This identity shows that, if the quotient $x - 3$ in the second step in (38) is itself divided by $x - 2$ until a constant remainder is obtained, then this remainder is the required value of c_1 and the quotient is c_0. If this division is accomplished synthetically, it may be combined with (38) in the table

$$\begin{array}{rrrr|l}
1 & -7 & 14 & -7 & \underline{2} \\
 & 2 & -10 & 8 & \\
\hline
1 & -5 & 4 & 1 & \\
 & 2 & -6 & & \\
\hline
1 & -3 & -2 & & \\
 & 2 & & & \\
\hline
1 & -1 & & &
\end{array} \tag{40}$$

The entry -1 in the seventh row and second column of (40) is c_1. The entry 1 in the seventh row and first column of (40) is c_0. These values of c_0, c_1, c_2, c_3 show that $U(u) = 0$ is indeed (31).

It will now be explained how to find the root u of (31) which, by (30), will yield the root x of (29) which is between 2 and 3. By (30) it follows that $2 < u + 2 < 3$ and $0 < u < 1$. Therefore u is a positive proper fraction. Hence u^3 is smaller than u, and u^2 is smaller than u. Therefore an approximate value of u is obtained by disregarding the terms involving the third and second powers of u in (31). Thus, an approximate value of u is obtained by solving

$$-2u + 1 = 0. \tag{41}$$

The notation $u =_a \frac{1}{2}$ will be used to indicate that the value $\frac{1}{2}$ obtained from (41) is merely an approximate value of u.

Since u is approximately $\frac{1}{2}$, the value of $U(0.5)$ is computed. Thus

$$\begin{array}{rrrr|l}
1 & -1.0 & -2.00 & 1.000 & \underline{0.5} \\
 & 0.5 & -0.25 & -1.125 & \\
\hline
1 & -0.5 & -2.25 & -0.125 &
\end{array}$$

Since $U(0) > 0$ and $U(0.5) < 0$, the root u of (31) is between 0 and 0.5. By synthetic substitution it is found that $U(0.4) > 0$.

Therefore

$$0.4 < u < 0.5 \tag{42}$$

and by (30)

$$2.4 < x < 2.5 \tag{43}$$

Thus it is known that 2.4 is an approximation to the required root of (29) and that this approximation is correct in the tenths place. By definition the statement that the root is *correct in k decimal places* means that the digits to the left of and in the kth decimal place are correct.

The digit which is in the second decimal place of u and hence also of x will now be determined by finding an equation which is related to (31) as (31) was related to (29). Thus the root u of (31) will be found by finding a number v such that

$$u = 0.4 + v \tag{44}$$

If $V(v)$ is the polynomial obtained by using (44) in (31), then v is a root of $V(v) = 0$. The coefficients of $V(v)$ will now be found from the coefficients of $U(u)$ by a table similar to (40) in which the coefficients of $U(u)$ were found from the coefficients of $f(x)$. The table is

$$\begin{array}{rrrr|l}
1 & -1.0 & -2.00 & 1.000 & \underline{0.4} \\
 & 0.4 & 0.24 & -0.896 & \\
\hline
1 & -0.6 & 2.24 & 0.104 & \\
 & 0.4 & -0.08 & & \\
\hline
1 & -0.2 & 2.32 & & \\
 & 0.4 & & & \\
\hline
1 & 0.2 & & &
\end{array} \tag{45}$$

Therefore v satisfies the equation

$$v^3 + 0.2v^2 - 2.32v + 0.104 = 0 \tag{46}$$

Since $0.4 < u < 0.5$ therefore $0.4 < 0.4 + v < 0.5$ and $0 < v < 0.1$. Also $v =_a 0.104/2.32$. Therefore $v =_a 0.04$. By synthetic substitution it is found that $V(0.04) > 0$ and $V(0.05) < 0$. Therefore

$$0.04 < v < 0.05 \tag{47}$$

and

$$(48) \qquad 2.44 < x < 2.45.$$

It is now known that 2.44 is an approximation to the required root of (29) and that this approximation is correct in the hundredths' place.

The digit which is in the third decimal place of v, and hence also of u and of x, will now be determined by finding an equation which is related to (46) as (46) was related to (31) and as (31) was related to (29). Thus, the root v will be known if a number r can be found such that

$$(49) \qquad v = 0.04 + r.$$

If $R(r)$ is the polynomial obtained by using (49) in (46), then r is a root of $R(r) = 0$. The table

$$(50) \qquad \begin{array}{rrrr|l}
1 & 0.20 & -2.3200 & 0.104000 & \underline{0.04} \\
 & 0.04 & 0.0096 & -0.092416 & \\
\hline
1 & 0.24 & -2.3104 & 0.011584 & \\
 & 0.04 & 0.0112 & & \\
\hline
1 & 0.28 & -2.2992 & & \\
 & 0.04 & & & \\
\hline
1 & 0.32 & & &
\end{array}$$

determines the coefficients of $R(r)$ from those of $V(v)$. Therefore r satisfies the equation

$$(51) \qquad r^3 + 0.32r^2 - 2.2992r + 0.011584 = 0.$$

By (49) and (47) it is true that $0 < r < 0.01$. Therefore $r =_a 0.011584/2.2992$, and $r =_a 0.005$. By synthetic substitution it is found that $V(0.005) > 0$, and $V(0.006) < 0$. Therefore

$$(52) \qquad 0.005 < r < 0.006,$$

and

$$(53) \qquad 2.445 < x < 2.446.$$

Therefore the approximation 2.445 to x is correct in three decimal places.

The number r will be known if a number s can be found such that

(54) $$r = 0.005 + s$$

The coefficients of the equation satisfied by s are determined by the appropriate tabulation, and it is found that the number s satisfies the equation

(55) $$s^3 + 0.335s^2 - 2.295925s + 0.000096125 = 0$$

The polynomial in (55) is designated by $S(s)$. By the linear terms in (55)

(56) $$s =_a \frac{0.000096125}{2.295925}$$

Therefore $s =_a 0.00004$. This indicates that the digit in the fourth decimal place is zero. This fact is verified by computing $S(0)$ and $S(0.0001)$. Since $S(0) > 0$ and $S(0.0001) < 0$, therefore the root s of (55) is indeed between 0 and 0.0001. Therefore

(57) $$2.4450 < x < 2.4451,$$

and the approximation 2.4450 to x is correct in four decimal places. The discussion preceding (57) illustrates the procedure if at any step the linear terms seem to yield a zero as the next digit.

It is to be noted especially that in this illustration of computation by Horner's method each of the roots x, u, v, r, s is a positive number. Therefore the various continued inequalities, which exhibit the closeness of approximation at each step, present no difficulties.

If a negative root of an equation is to be computed, this simplicity may also be achieved. The method will now be explained. The equation $x^3 - x^2 - x + 2 = 0$ has a root between -2 and -1, because, if $f(x) \equiv x^3 - x^2 - x + 2$, then $f(-2) < 0$ and $f(-1) > 0$. By $x = -z$, the equation $-z^3 - z^2 + z + 2 = 0$ is obtained. An equivalent equation is $z^3 + z^2 - z - 2 = 0$. This equation has a root between 1 and 2. The negative of this root is the root between -2 and -1 of $x^3 - x^2 - x + 2 = 0$. Horner's method is applied to compute the root of $z^3 + z^2 - z - 2 = 0$ which is between 1 and 2.

PROBLEMS

For each equation in the preceding set of problems find by Horner's method an approximation to each real root correct in three decimal places.

The preceding method of obtaining the digits in the successive decimal places of a root of an equation is referred to as the method of transformed equations. It could be continued until the desired number of decimal places had been reached. There is a more simple method which may be used advantageously at any step after three decimal places have been obtained. By this new method about as many more decimal places are obtained simultaneously as have already been obtained. The new method, which is referred to as the correction method, will now be explained.

By synthetic substitution it is found that $S(0.00004) > 0$ and $S(0.00005) < 0$. Therefore

(58) $$0.00004 < s < 0.00005,$$

and

(59) $$2.44504 < x < 2.44505.$$

The approximation 2.44504 is correct in five decimal places.

It will now be explained how (58) can be used to obtain an approximation to s correct in eight decimal places. Since $0.00004 < s$, it is true that $(0.00004)^3 < s^3$, and $0.335(0.00004)^2 < 0.335s^2$. Hence $(0.00004)^3 + 0.335(0.00004)^2 < s^3 + 0.335s^2$. The number $(0.00004)^3 + 0.335(0.00004)^2$ will be designated by C_1 and will be computed later. At present the details are more simply expressed if C_1 is used instead of this number. Therefore,

(60) $$C_1 = (0.00004)^3 + 0.335(0.00004)^2,$$

(61) $$C_1 < s^3 + 0.335s^2.$$

If the equation (55) is rewritten in the form $0 = s^3 + 0.335s^2 - 2.295925s + 0.000096125$, then the inequality (61) can be subtracted from the equation correctly. The result is

(62) $$0 - C_1 > -2.295925s + 0.000096125.$$

Multiplication of both sides of (62) by -1 gives

(63) $$C_1 < 2.295925s - 0.000096125.$$

Addition of 0.000096125 to each side of (63) gives

(64) $$C_1 + 0.000096125 < 2.295925s.$$

Division of both sides of (64) by 2.295925 gives

(65) $$\frac{C_1 + 0.000096125}{2.295925} < s.$$

Comparison of this result with the fraction in (56), from which the approximation 0.00004 was obtained, shows that the numerator of (56) is too small and that a correction of more than C_1 should be added to this numerator to obtain the exact value of s.

Again since $s < 0.00005$ by (58), it is true that $s^3 < (0.00005)^3$ and $0.335s^2 < 0.335(0.00005)^2$. Hence $s^3 + 0.335s^2 < (0.00005)^3 + 0.335(0.00005)^2$. The number $(0.00005)^3 + 0.335(0.00005)^2$ will be designated by C_2. Therefore,

$$C_2 = (0.00005)^3 + 0.335(0.00005)^2, \tag{66}$$

$$s^3 + 0.335s^2 < C_2. \tag{67}$$

The result of subtraction of (67) from (55) is

$$-2.295925s + 0.000096125 > 0 - C_2. \tag{68}$$

Therefore

$$2.295925s - 0.000096125 < C_2, \tag{69}$$

$$2.295925s < C_2 + 0.000096125, \tag{70}$$

$$s < \frac{C_2 + 0.000096125}{2.295925}. \tag{71}$$

The exact value of s is between the fractions in (65) and (71).

The numbers C_1 and C_2 will be computed first. Then the numerators in (65) and (71) will be found. Finally the two divisions will be performed. The quotients will be numbers between which s lies. It will be found that these quotients agree in eight decimal places and disagree in the ninth decimal place. Thus s will have been found correct in eight decimal places. Finally, a single division by which the two longer divisions may be replaced will be explained.

By (60) C_1 is the value of the polynomial

$$z^3 + 0.335z^2 + 0\,z + 0 \tag{72}$$

when z is replaced by 0.00004. The synthetic substitution for the calculation of C_1 is exhibited now.

1	0.33500	0.00000 00000	0.00000 00000 00000	0.00004
	0.00004	0.00001 34016	0.00000 00005 36064	
1	0.33504	0.00001 34016	0.00000 00005 36064	

Therefore

$$C_1 = 0.000000000536064. \tag{73}$$

The value of C_2 is obtained by synthetic substitution of 0.00005 in (72). Therefore

$$C_2 = 0.000000000837625. \tag{74}$$

Substitution of (73) in (65) and (74) in (71) shows that

$$\frac{0.000096125536064}{2.295925} < s < \frac{0.000096125837625}{2.295925}. \tag{75}$$

These divisions will now be exhibited

```
                       0 000 041 867
2 295 925 | 0 000 096 125 536 064
                  91 837 00
                  ---------
                   4 288 536
                   2 295 925
                   ---------
                   1 992 611 0
                   1 836 740 0
                   -----------
                     155 871 06
                     137 755 50
                     ----------
                      18 115 564
                      16 071 475
                      ----------
                       2 014 089

                       0 000 041 868
2 295 925 | 0 000 096 125 837 625
                  91 837 00
                  ---------
                   4 288 837
                   2 295 925
                   ---------
                   1 992 912 6
                   1 836 740 0
                   -----------
                     156 172 62
                     137 755 50
                     ----------
                      18 417 125
                      18 367 400
                      ----------
                          49 725
```

These quotients agree in eight decimal places and disagree in the ninth Therefore

(76) $$0\ 000041867 < s < 0\ 000041868,$$

(77) $$2\ 445041867 < x < 2\ 445041868$$

Therefore the approximation 2 44504186 for x is correct in eight decimal places

In the following contracted division the dividend is smaller than the dividend on the left-hand side of (75) Also, as much as possible is carried, and hence each partial dividend is as small as possible Therefore the quotient is less than the exact value of s In the first step

```
     *           0 00 004
2 295 925 | 0 000 096 125
                   91 837
                   ------
                    4 288
```

of the contracted division the * over the 9 in the fourth decimal place of the divisor indicates that only this much of the original divisor is used The second line is 91837 instead of 91836 because 'd have been carried if the complete divisor had been used second step

```
   * *           0 00 0041
2 295 925 | 0 000 096 125
                   91 837
                   ------
                    4 288
                    2 296
                    -----
                    1 992
```

the * over the 5 in the third decimal place of the divisor indicates that only this much of the original divisor is used The entry 2296 in the fourth line appears instead of 2295 to insure that the next partial dividend will be as small as possible At each step an * is placed over the next digit to the left in the divisor, and only that much is used as a divisor The complete contracted division is

```
      *** *          0.00 001 186 7
2.205 925 ) 0.000 096 125
                   91 837
                   ------
                    4 288
                    2 206
                   ------
                    1 992
                    1 837
                   ------
                      155
                      138
                      ---
                       17
                       16
                       --
```

A similar contracted division could be carried out for the second of the original divisions. Its contracted dividend would be 0.000096126. At each step as little as possible would be carried. Each partial dividend would be as large as possible. Therefore the quotient would be greater than the exact value of s. The two contracted divisions would differ only in the last column. No difference in the last column can affect the 6 in the eighth decimal place of the quotient. Therefore the approximation 0.00001186 for s is correct in eight decimal places. The approximation 2.44501186 for x is correct in eight decimal places. The exact value of x is nearer 2.44501187.

The minimum computation which may be displayed in Horner's method is the tabulations such as (40), (45), and (50), the synthetic substitutions to obtain C_1 and C_2, and a contracted division. The auxiliary computations and inequalities, which prove that the decimal approximation so obtained is correct in the number of decimal places asserted, may be exhibited to advantage.

The references give other methods of computing a real root in decimal form.

PROBLEMS

For each equation on page 93 find an approximation to each real root correct in six decimal places.

CHAPTER 5

INTRODUCTION TO DETERMINANTS

1 Systems of linear equations and determinants In this chapter some methods of solving systems of linear equations will be illustrated by means of equations in three unknowns In a general *system of simultaneous linear equations there may be* any number of equations and any number of unknowns If the coefficients in the equations are numbers and if there are only a few equations and only a few unknowns there are several simple methods of finding whether there is a set of values of the unknowns which satisfy all the equations These methods also give all such sets of values If the equations have literal coefficients or if there are many equations or many unknowns these methods may lead to very complicated results However the method of determinants and matrices leads to very simple results This simplicity is achieved only after an exhaustive study of the meaning and properties of determinants and matrices

2 Solution of certain systems of numerical equations in three unknowns A method of solving one equation in three unknowns will now be illustrated by means of the particular equation

(1) $$2x - y + 5z = -1$$

The ordered set of numbers 1 3 0 is a solution of (1) because $2\ 1 - 3 + 5\ 0 = -1$ In general if a b c is an ordered set of numbers the statement that this set is a solution of (1) means that $2a - b + 5c = -1$ This fact is also expressed by the statement *that* a b c *satisfy* (1)

A method of finding all solutions of (1) will now be explained If (1) is solved for y in terms of x and z the result is

(2) $$y \quad 2x + 5z + 1$$

If the value 0 is assigned to x and the value 1 to z then $y = 6$ Again if -1 and 2 are assigned to x and z respectively then $y = 9$

In general, if arbitrary values are assigned to x and z in (2), then as many solutions of (1) as desired can be found. Now an ordered triple of numbers which satisfy (2) is an ordered triple of numbers which satisfy (1). Conversely, an ordered triple of numbers which satisfy (1) is an ordered triple of numbers which satisfy (2). Therefore (1) and (2) are equivalent. It is also said that (2) *gives the general solution of (1) for y.* All solutions of (1) are found by the method of using arbitrary values for x and z in (2). Since each of the quantities x and z takes infinitely many values independently of the other, it is said that there is *a double infinity of solutions of* (1).

It is to be noted especially that (2) expresses *y as a linear function of x and z.* This function is *a non-homogeneous function of these variables* because there is a term which involves neither x nor z.

A method of solving a system of two linear equations in three unknowns will now be illustrated by means of the particular equations

$$(3)\qquad \begin{aligned} x + 2y - 3z &= 2, \\ 2x - y + z &= -3. \end{aligned}$$

The second equation in (3) is equivalent to the equation obtained from it by multiplying both sides by 2. Hence (3) are equivalent to

$$(4)\qquad \begin{aligned} x + 2y - 3z &= 2, \\ 4x - 2y + 2z &= -6. \end{aligned}$$

Again, the set (4) is equivalent to the set (5) obtained by using the first equation in (4) as the first equation in (5), and the sum of the two equations in (4) as the second equation in (5). Hence (3) are equivalent to

$$(5)\qquad \begin{aligned} x + 2y - 3z &= 2, \\ 5x \qquad - z &= -4. \end{aligned}$$

Now (5) are equivalent to

$$(6)\qquad \begin{aligned} z &= 5x + 4, \\ y &= \frac{2 - x + 3z}{2}, \end{aligned}$$

and hence to

$$(7)\qquad \begin{aligned} z &= 5x + 4, \\ y &= 7x + 7. \end{aligned}$$

The particular solution of (3) which is obtained from (7) by assigning the value 0 to x is 0, 7, 4. Another particular solution is 1, 14, 9. Since exactly one quantity, namely x, in (7) may be assigned infinitely many values, there is *a single infinity of solutions of* (3). It is to be noted especially that equations (3) have been solved for y and z in terms of x, and that *the general solution* (7) *expresses each of y and z as a linear, non-homogeneous function of x.* Equations (3) could have been solved for x and y in terms of z, or for x and z in terms of y. Each of these methods gives all solutions of (3), by assigning arbitrary values to the transposed variables.

An important fact about some systems of two linear equations in three unknowns is illustrated by the equations

$$(8)\qquad \begin{aligned} x + 2y - 3z &= 2, \\ -3x - 6y + 9z &= 7. \end{aligned}$$

The first equation in (8) will be designated by (8_1), and the second by (8_2). A particular solution of (8_1) is 1, 2, 1. This set is not a solution of (8_2) because $-3 \cdot 1 - 6 \cdot 2 + 9 \cdot 1 \neq 7$. It will now be proved that there is no triple of numbers such that each equation in (8) is satisfied by the triple. This will be done by showing that, if a, b, c are three numbers which satisfy (8), then there is a contradiction. If a, b, c satisfy (8), then $a + 2b - 3c = 2$, and $-3a - 6b + 9c = 7$. Multiplication of the first of these equations by -3 shows that $-3a - 6b + 9c = -6$. Since $-6 \neq 7$, there is a contradiction of the second equation. This system (8) is an illustration of a system which has no solution. By definition *a system of equations is inconsistent* if there is no solution of the system. *The equations are said to be inconsistent.*

A method of solving a system of three linear equations in three unknowns will now be illustrated by means of

$$(9)\qquad \begin{aligned} x + 2y - 3z &= 2, \\ 2x - y + z &= -3, \\ 6x - y + z &= 1. \end{aligned}$$

Since the first two equations in (9) are precisely the equations (3), equations (9) are equivalent to

$$\begin{aligned} z &= 5x + 4, \\ y &= 7x + 7, \\ 6x - y + z &= 1. \end{aligned} \tag{10}$$

Hence (9) are equivalent to

$$\begin{aligned} z &= 5x + 4, \\ y &= 7x + 7, \\ 6x - (7x + 7) + (5x + 4) &= 1. \end{aligned} \tag{11}$$

Hence (9) are equivalent to

$$x = 1, \quad y = 7 \cdot 1 + 7, \quad z = 5 \cdot 1 + 4. \tag{12}$$

Hence there is one and only one solution of (9), namely 1, 14, 9. It is said that equations (9) have a *unique solution*.

The system of three equations formed by adjoining to the system (3) the equation $2x + 9y - 13z = 11$ is an illustration of a system of three equations with a single infinity of solutions. This is true because, if (7) are substituted in $2x + 9y - 13z = 11$, the result is $2x + 9(7x + 7) - 13(5x + 4) = 11$. This last equation is true for all values of x. Hence $2x + 9y - 13z = 11$ is satisfied by all solutions of (3).

The system of three equations which is formed by adjoining to the system (3) the equation $-3x - 6y + 9z = 7$ is inconsistent because (8) are two of these equations and it has been proved that (8) are inconsistent.

These illustrations show that it is not the number q of equations which determines whether the equations are inconsistent, and, if they are consistent, how many solutions there are. Later it will be explained precisely how these facts are determined by the coefficients of the variables and by the constants in the equations.

Each of the equations in the system

$$\begin{aligned} 3x - 2y + z &= 0, \\ x + y - z &= 0, \end{aligned} \tag{13}$$

is a *homogeneous equation* since the constant in the equation is zero Now (13) are equivalent to

$$\begin{aligned} -2y + z &= -3x \\ y - z &= -x \end{aligned} \tag{14}$$

and hence to

$$\begin{aligned} -y &= -4x \\ y - z &= -x \end{aligned} \tag{15}$$

Therefore (13) are equivalent to

$$\begin{aligned} y &= 4x \\ z &= 5x \end{aligned} \tag{16}$$

These equations give *the general solution of (13)* From them as many numerical solutions as desired can be found Thus 0 0 0 is a solution 1 4 5 is a solution -2 -8 -10 is a solution The solution 0 0 0 is called *the zero solution* It is also called the *trivial solution* It is to be noted especially that (16) expresses y *as a linear homogeneous function of* x *and* z *as a linear homogeneous function of* x In this respect the solution (16) of the homogeneous equations (13) is to be contrasted with the solution (7) of the non homogeneous equations (3) The equations (13) have a single infinity of solutions

If $x \neq 0$ equations (16) can be written in the form

$$\begin{aligned} \frac{y}{x} &= \frac{4}{1} \\ \frac{z}{x} &= \frac{5}{1} \end{aligned} \tag{17}$$

Thus equations (13) have been *solved for the ratios* y/x and z/x Another way of writing (17) is

$$\begin{aligned} \frac{y}{4} &= \frac{x}{1} \\ \frac{z}{5} &= \frac{x}{1} \end{aligned} \tag{18}$$

Hence the solution of (13) can be written

$$\frac{x}{1} = \frac{y}{4} = \frac{z}{5}. \tag{19}$$

The statement that

$$x:y:z = 1:4:5 \tag{20}$$

means, by definition, precisely (19).

The second illustration of homogeneous equations is

$$\begin{aligned} x - y + 2z &= 0, \\ 2x + y + z &= 0, \\ -2x + y + 2z &= 0. \end{aligned} \tag{21}$$

These equations are equivalent to

$$\begin{aligned} x - y + 2z &= 0, \\ 3x \phantom{{}-y} + 3z &= 0, \\ -x \phantom{{}-y} + 4z &= 0, \end{aligned} \tag{22}$$

and hence to

$$\begin{aligned} x - y + 2z &= 0, \\ x \phantom{{}-y} + z &= 0, \\ x \phantom{{}-y} - 4z &= 0. \end{aligned} \tag{23}$$

Hence (21) are equivalent to

$$\begin{aligned} x - y + 2z &= 0, \\ 5z &= 0, \\ x \phantom{{}-y} - 4z &= 0. \end{aligned} \tag{24}$$

Thus (21) are equivalent to

$$x = 0, \quad y = 0, \quad z = 0. \tag{25}$$

Thus (21) is an illustration of a system of homogeneous equations for which the zero solution is the only solution.

There are three rules which may be used in the discussion of a system of numerical linear equations. These rules were followed in each of the preceding illustrative examples. Thus, in the discussion of (3) there is a sequence of equivalent systems (3), (4), (5), (6), and (7). The systems in this sequence illustrate *the first*

rule which is that the number of equations in each system of the sequence is the number of equations in the original system

It will now be explained how the sequence (3) (7) also illustrates the second and third rules Thus (4) is obtained from (3) by multiplying (3_1) by the constant 1 and (3_2) by the constant 2 Then (5_1) is (4_1) and (5_2) is obtained by using (4_1) in (4_2) to eliminate y Again (6) is obtained from (5) by multiplying (5_2) by the constant 1 and (5_1) by the constant $\frac{1}{2}$ Then (7_1) is (6_1) and (7_2) is obtained by using (6_1) in (6_2) to eliminate z *The second rule* is that each equation in a system may be replaced by a non zero constant multiple of itself *The third rule* is that one equation is used in each of the other equations to eliminate from these other equations a selected (fixed) variable Often the second and third rules are used simultaneously

If the first rule is always observed and if the second and third rules are used to eliminate variables in turn without introducing again those already eliminated then either a solution of the original system or a contradiction is obtained If a contradiction is obtained the original equations are inconsistent

PROBLEMS

For each of the following systems show whether the equations in the system are consistent or inconsistent If the equations are consistent show that there is a unique solution or find the general solution and state how many solutions the system has

1 $$\begin{aligned} 3x - y + 7z &= 2 \\ x + y - 2z &= 1 \\ 5x + 3y + 4z &= 3 \end{aligned}$$

2 $$\begin{aligned} 3x + 4y + 4z &= 15 \\ x + 4y + 2z &= 7 \\ x + 20y + 6z &= 19 \end{aligned}$$

3 $$\begin{aligned} x + y - 2z &= 7 \\ 4x - 2y + z &= -11 \\ 3x + y - 3z &= 8 \end{aligned}$$

4 $$\begin{aligned} v + 2s + t &= 9 \\ 2v + 5s - 2t &= -1 \\ -3v - 7s + 4t &= 2 \end{aligned}$$

5 $$\begin{aligned} 4u + 2v + 2w &= 0 \\ u + v - w &= 0 \end{aligned}$$

6 $$\begin{aligned} 7v - 7s - t &= 0 \\ 10v - s - 4t &= 0 \end{aligned}$$

7 $$\begin{aligned} 3v - 5s + t &= -3 \\ v + 2s - 7t &= 1 \end{aligned}$$

8 $$\begin{aligned} u - 5v + 3w &= 0 \\ -2u + 3v + w &= 7 \\ 3u - 7v + w &= 1 \end{aligned}$$

9 $$\begin{aligned} 3x + 2y - z &= -7 \\ x + y - z &= -2 \\ x - 3y + 7z &= 2 \end{aligned}$$

10 $$\begin{aligned} 2x - 5y - 3z &= 1 \\ 7x + 2y + 4z &= -1 \\ -x + 3y - 2z &= 11 \\ 3x + y + z &= 2 \end{aligned}$$

11. $\begin{aligned} u - 2v + 3w &= 1, \\ u &= 2v + w + 1. \end{aligned}$

12. $\begin{aligned} 3u + v + 2w &= 2, \\ 2u + 7v - 5w &= 14. \end{aligned}$

13. $\begin{aligned} v + s - 3t &= 7, \\ 2v + t &= 5s, \\ v + 5s - 7t &= 15, \\ v - 4s + 2t &= -3. \end{aligned}$

14. $\begin{aligned} -2x + 5y + z &= 12, \\ -x + 2y + z &= 5, \\ 4x + y - 13z &= -2, \\ x + y - 4z &= 1. \end{aligned}$

3. Systems of three linear equations in three unknowns. Determinants of order three. Determinants of order two. Matrices. In this section systems of three linear equations in three unknowns are considered. The equations have literal rather than numerical coefficients. The equations are given the notation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= k_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= k_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= k_3. \end{aligned} \tag{26}$$

If $k_1 = 0$, $k_2 = 0$, $k_3 = 0$, then the equations are homogeneous.

In (26) it is assumed that there are indeed three variables in the equations and three equations in the system. This is accomplished by the assumption that at least one of the numbers a_{11}, a_{21}, a_{31} is not zero, that at least one of a_{12}, a_{22}, a_{32} is not zero, and that at least one of a_{13}, a_{23}, a_{33} is not zero, and by the assumption that at least one of a_{11}, a_{12}, a_{13} is not zero, that at least one of a_{21}, a_{22}, a_{23} is not zero, and that at least one of a_{31}, a_{32}, a_{33} is not zero.

There are four parts to the discussion. In part I it is assumed that there is a solution of (26). Then certain results are obtained in succession. These results are necessary conditions for the existence of a solution of (26) because they follow from the hypothesis that there is a solution. The fundamental definition of a determinant is illustrated in part I. The discussion in part I is preliminary to the proofs of theorems 1 and 2 in part II. The proof of theorem 3, which constitutes part III, is based on the discussion in part I. In part IV the discussion of the possible cases in the solution of equations (26) is completed. Some of these results are not proved. They are merely stated as illustrations of general theorems to be proved later. These statements are simplified by the use of the new idea of matrix.

In part I it is assumed that there is a solution of (26). Thus there are three numbers c_1, c_2, c_3 such that

$$
\begin{aligned}
(27)\qquad & a_{11}c_1 + a_{12}c_2 + a_{13}c_3 = k_1, \\
& a_{21}c_1 + a_{22}c_2 + a_{23}c_3 = k_2, \\
& a_{31}c_1 + a_{32}c_2 + a_{33}c_3 = k_3.
\end{aligned}
$$

The distinction between (26) and (27) is to be noted especially. As explained in section 1 of chapter 1, equations (27) state that the ordered set c_1, c_2, c_3 is a solution of the equations (26). In part I it is customary to use (26) instead of (27) with the understanding that x_1, x_2, x_3 temporarily mean values of the unknowns which satisfy the equations.

In part I equations (26) will be used instead of (27), with this understanding. Thus in part I the left-hand side of (26_1) is a number and this number is indeed the number k_1. Then it follows that

$$(28)\qquad a_{11}a_{23}x_1 + a_{12}a_{23}x_2 + a_{13}a_{23}x_3 = k_1a_{23}$$

is true. Similarly from (26_2) it follows that

$$(29)\qquad a_{21}a_{13}x_1 + a_{22}a_{13}x_2 + a_{23}a_{13}x_3 = k_2a_{13}.$$

Now by subtraction of (29) from (28) there follows the equation

$$(30)\quad (a_{11}a_{23} - a_{21}a_{13})x_1 + (a_{12}a_{23} - a_{22}a_{13})x_2 = (k_1a_{23} - k_2a_{13}).$$

Since (30) results from the hypothesis that there is a solution of (26), (30) is *a necessary condition for the existence of a solution of (26)*. Similarly if (26_1) is multiplied by a_{33} and from this product there is subtracted the result of multiplying (26_3) by a_{13}, there is obtained the necessary condition

$$(31)\quad (a_{11}a_{33} - a_{31}a_{13})x_1 + (a_{12}a_{33} - a_{32}a_{13})x_2 = (k_1a_{33} - k_3a_{13}).$$

In (30) the complicated coefficient of x_1 will be designated by b_{11}, the coefficient of x_2 by b_{12}, and the constant term on the right by d_1. Thus

$$
\begin{aligned}
& b_{11} = a_{11}a_{23} - a_{21}a_{13}, \\
(32)\qquad & b_{12} = a_{12}a_{23} - a_{22}a_{13}, \\
& d_1 = k_1a_{23} - k_2a_{13}.
\end{aligned}
$$

Then (30) becomes the more simple equation

$$b_{11}x_1 + b_{12}x_2 = d_1.$$

Similarly, by introducing the notations

$$(33)\qquad \begin{aligned} b_{21} &= a_{11}a_{33} - a_{31}a_{13}, \\ b_{22} &= a_{12}a_{33} - a_{32}a_{13}, \\ d_2 &= k_1a_{33} - k_3a_{13}, \end{aligned}$$

equation (31) becomes the more simple equation

$$b_{21}x_1 + b_{22}x_2 = d_2.$$

Thus, if (32) and (33) are used, then the two necessary conditions (30) and (31) become the more simple equations

$$(34)\qquad \begin{aligned} b_{11}x_1 + b_{12}x_2 &= d_1, \\ b_{21}x_1 + b_{22}x_2 &= d_2. \end{aligned}$$

A single necessary condition, involving only x_1, will now be obtained from (34). Thus, if (34_1) is multiplied by b_{22} and (34_2) by b_{12}, and if the latter result is subtracted from the former, then the result is

$$(35)\qquad (b_{11}b_{22} - b_{21}b_{12})x_1 = d_1b_{22} - d_2b_{12}.$$

This equation is expressed in terms of the original letters by using (32) and (33). In the result the coefficient of x_1 is

$$(36)\qquad (a_{11}a_{23} - a_{21}a_{13})(a_{12}a_{33} - a_{32}a_{13}) \\ - (a_{11}a_{33} - a_{31}a_{13})(a_{12}a_{23} - a_{22}a_{13}).$$

Also the constant term on the right-hand side of (35) becomes

$$(37)\qquad (k_1a_{23} - k_2a_{13})(a_{12}a_{33} - a_{32}a_{13}) \\ - (k_1a_{33} - k_3a_{13})(a_{12}a_{23} - a_{22}a_{13}).$$

If the products in (36) are expanded and the result is simplified, then (36) becomes

$$(38)\qquad a_{13}(a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} \\ - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}).$$

In a similar way it could be proved that (37) becomes (39) But this is more easily proved by noting that (37) may be obtained from (36) by replacing a_{11} by k_1, a_{21} by k_2, and a_{31} by k_3 If these replacements are made in (38), the result is (39) Hence (37) becomes

$$(39)\quad a_{13}(k_1a_{22}a_{33} - k_1a_{32}a_{23} + k_2a_{32}a_{13} - k_2a_{12}a_{33} + k_3a_{12}a_{23} - k_3a_{22}a_{13})$$

The second factor in (38) is so complicated that a new symbol is introduced to designate it The symbol is

$$(40)\quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

This symbol means, by definition, precisely the sum

$$(41)\quad a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}.$$

Two simple rules by which the expression (41) can be written down directly from the symbol (40) will be explained later The number (41) is called *a determinant* The symbol (40) is called *the symbol of the determinant (41)* The determinant and its symbol are of *order three* because there are three rows and three columns in (40) The symbol (40) may be called a determinant if no confusion results The nine numbers a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} from which the number (41) is formed are called *the elements of the determinant* The symbol (40) and the number (41) will be designated by D Since the nine elements in (40) are the coefficients of the unknowns in (26) and are ordered as those coefficients are ordered in (26), D is called *the determinant of the coefficients of (26)*

If a_{11}, a_{21}, a_{31} in (41) are replaced by k_1, k_2 k_3 respectively, the second factor in (39) is obtained Therefore the second factor in (39) is a determinant Its symbol is

$$(42)\quad \begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}$$

This determinant and the symbol (42) for it are designated by D_1,

for a reason which will be stated later. Thus D_1 is the number

$$(43) \quad k_1a_{22}a_{33} - k_1a_{32}a_{23} + k_2a_{32}a_{13} - k_2a_{12}a_{33} + k_3a_{12}a_{23} - k_3a_{22}a_{13}.$$

By (38), (39), and the notations D and D_1 for the numbers (41) and (43) respectively, the necessary condition (35) becomes

$$(44) \qquad a_{13}Dx_1 = a_{13}D_1.$$

In a similar manner two other necessary conditions involving x_1 are obtained. They are

$$(45) \qquad a_{23}Dx_1 = a_{23}D_1,$$

$$(46) \qquad a_{33}Dx_1 = a_{33}D_1.$$

Now, by the hypothesis which follows (26), either $a_{13} \neq 0$, or $a_{23} \neq 0$, or $a_{33} \neq 0$. Hence *the necessary condition*

$$(47) \qquad Dx_1 = D_1$$

is obtained from at least one of the conditions (44), (45), (46).

There is a necessary condition involving x_2 which is similar to (47), and there is a similar one for x_3. They are proved in the same way. To state them, D_2 and D_3 are defined by

$$(48) \quad D_2 = \begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}.$$

The symbol for D_2 is obtained from the symbol (40) for D by replacing a_{12}, a_{22}, a_{32} respectively by k_1, k_2, k_3; also the symbol for D_3 is obtained from (40) by replacing a_{13}, a_{23}, a_{33} respectively by k_1, k_2, k_3. If these replacements are made in (41), it is found that

$$(49) \quad D_2 = a_{11}k_2a_{33} - a_{11}k_3a_{23} + a_{21}k_3a_{13} - a_{21}k_1a_{33} + a_{31}k_1a_{23} - a_{31}k_2a_{13},$$

$$(50) \quad D_3 = a_{11}a_{22}k_3 - a_{11}a_{32}k_2 + a_{21}a_{32}k_1 - a_{21}a_{12}k_3 + a_{31}a_{12}k_2 - a_{31}a_{22}k_1.$$

The three necessary conditions are therefore

$$(51) \qquad Dx_1 = D_1, \quad Dx_2 = D_2, \quad Dx_3 = D_3.$$

The first rule by which the expression (41) can be written down directly from the symbol (40) is important because it is a simple illustration of the general definition of a determinant If n designates the number of rows and columns in the symbol of a determinant, then $n = 3$ in the determinant (40) First, there are 3! terms in (41) Next, the expression (41) is written in the form $(-1)^0a_{11}a_{22}a_{33} + (-1)^1a_{11}a_{32}a_{23} + (-1)^2a_{21}a_{32}a_{13} + (-1)^1a_{21}a_{12}a_{33} + (-1)^2a_{31}a_{12}a_{23} + (-1)^3a_{31}a_{22}a_{13}$ Therefore each term in (41) is a power of -1 multiplied by a literal product of three factors In each term the second subscripts are in the natural order The first subscripts in the six terms are 1, 2, 3, 1, 3 2, 2, 3, 1, 2, 1, 3, 3, 1, 2, 3, 2, 1 It is to be noted especially that these are precisely all *the arrangements of the numbers 1, 2, 3*

The correct exponent of the power of -1 by which each literal product is multiplied is determined when the second subscripts are in the natural order, by the particular arrangement which the first subscripts form The rule for determining the correct exponent will now be explained In the second term $(-1)^1a_{11}a_{32}a_{23}$ the first subscripts form the arrangement 132 In this arrangement the number 3 precedes the number 2, and 3 is larger than 2 This fact is also described by saying that in this arrangement there is one *inversion* due to the numbers 2 and 3 Since 1 is smaller than each of the numbers 3 and 2 which 1 precedes, there is no other inversion in this arrangement Therefore the number of inversions in the arrangement 132 is 1 This number 1 of inversions is the exponent of the power of -1 by which the literal product $a_{11}a_{32}a_{23}$ is multiplied to obtain the term $(-1)^1a_{11}a_{32}a_{23}$ Again, in the third term $(-1)^2a_{21}a_{32}a_{13}$ the first subscripts form the arrangement 231 In this arrangement there are 2 inversions This number 2 of inversions is the exponent of the power of -1 by which the literal product $a_{21}a_{32}a_{13}$ is multiplied to obtain the term $(-1)^2a_{21}a_{32}a_{13}$ Similarly, the exponent 0 in the first term is the number of inversions in the arrangement 123 of first subscripts The general rule is that the exponent of the power of -1 by which the literal product is multiplied is the *number of inversions in the arrangement of first subscripts, when the second subscripts are in the natural order*

Now the first rule for writing down the expression (41) can be stated simply The determinant (41) of the third order is the sum of 3! terms Each term is a power of -1 multiplied by a literal

product in which the second subscripts are in the natural order. The exponent of the power of -1 is the number of inversions in the arrangement formed by the first subscripts. This rule is merely a restatement of (41). Therefore this rule is a definition of the determinant of the third order whose symbol is (40). This rule is stated because of its theoretical importance. In practice the following rule is used.

The second rule for writing down the expression (41) directly from the symbol (40) is easily stated if determinants of order two are defined. Determinants of order two could be introduced in solving two linear equations in two unknowns. This is not done because determinants of order two are not complicated expressions and no simplification results when they are used in solving two linear equations in two unknowns. On the other hand, determinants of order two simplify the practical use of the definition (41) of the determinant whose symbol is (40). If a, b, c, and d are letters which represent numbers, then *the determinant of order two*, whose symbol is

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix}, \tag{52}$$

is the number

$$ad - bc. \tag{53}$$

Now (41) can be written in the form $a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13})$. Hence (41) is in fact

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}. \tag{54}$$

In this expression the multipliers a_{11}, a_{21}, a_{31} are the elements in the first column of (40) and the signs alternate. The determinant of order two which is multiplied by a_{11} in (54) is obtained from the symbol (40) by deleting from (40) the row and column in which a_{11} stands. The determinant of order two which is multiplied by a_{21} in (54) is obtained from the symbol (40) by deleting from (40) the row and column in which a_{21} stands. A similar statement is true for the determinant which is multiplied by a_{31} in (54). The expression (54) gives a practical rule for writing down the number (41) from the symbol (40).

Other facts about determinants of order three will appear as special cases of rules for determinants of order n.

PROBLEMS

1 Evaluate each of the determinants $\begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix}$ $\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}$ $\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}$

2 Evaluate the determinant $\begin{vmatrix} 2 & 1 & 3 \\ 5 & 2 & 2 \\ 1 & 1 & 4 \end{vmatrix}$ using (54) and the results of problem 1

3 Evaluate each of the determinants

$$\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} \quad \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

4 Evaluate the determinant $\begin{vmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \\ 2 & -1 & 1 \end{vmatrix}$ using (54) and the results of problem 3

5 Using (54) show that $\begin{vmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 4 & 1 & 1 \end{vmatrix}$ is the function $2x + 3y - 11$ Hence show that $\begin{vmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 4 & 1 & 1 \end{vmatrix} = 0$ is an equation of the straight line which passes through the points (1 3) and (4 1)

6 Show that if $(a_1\ b_1)$ and $(a_2\ b_2)$ are two distinct points then

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$$

is an equation of the straight line which passes through these points If $(a_3\ b_3)$ is a point distinct from each of these points and if these three points are collinear then $\begin{vmatrix} a_3 & b_3 & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$ Prove that if this determinant is zero then these three points are collinear

7 Show that $\begin{vmatrix} x & y & 1 \\ a & b & 1 \\ 1 & m & 0 \end{vmatrix}$ is the function $-mx + y + am - b$ Hence show that an equation of the straight line which passes through the point $(a\ b)$ and has slope m is $\begin{vmatrix} x & y & 1 \\ a & b & 1 \\ 1 & m & 0 \end{vmatrix} = 0$

8 Let $(a_1\ b_1)$ $(a_2\ b_2)$ $(a_3\ b_3)$ be three distinct points Prove that the area of the triangle with vertices at these points is one-half or the negative of one-half of the number $\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

Theorem 1 and theorem 2 constitute part II.

THEOREM 1. *Let D be the determinant of the coefficients of the three variables in the three linear equations (26), and let D_1, D_2, D_3 be defined by (42) and (48). If $D \neq 0$ and if there is a solution of these equations, then that solution is the ordered triple D_1/D, D_2/D, D_3/D.*

PROOF. If (26) have a solution and if $D \neq 0$, the necessary conditions (51) imply that

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}. \tag{55}$$

THEOREM 2. *Let D be the determinant of the coefficients of the three variables in the three linear equations (26), and let D_1, D_2, D_3 be defined by (42) and (48) If $D \neq 0$, then D_1/D, D_2/D, D_3/D are numbers, and this ordered triple is a solution of (26).*

It is to be proved that $a_{11}(D_1/D) + a_{12}(D_2/D) + a_{13}(D_3/D) = k_1$. This will follow if it is proved that

$$a_{11}D_1 + a_{12}D_2 + a_{13}D_3 = k_1 D. \tag{56}$$

If the expressions (43), (49), (50), and (41) are used for D_1, D_2, D_3, and D respectively, and the products and sums indicated in (56) are computed, it is found that (56) is true. In the same way it is proved that this triple is a solution of the other two equations in (26). It is to be noted that the hypothesis of part I is not used in this proof. This completes the proof of theorem 2.

THEOREM 3. *Let D be the determinant of the coefficients of the three variables in the three linear equations (26), and let D_1, D_2, D_3 be defined by (42) and (48). If $D = 0$ and at least one of the numbers D_1, D_2, D_3 is not zero, then there is no solution of (26), and (26) are inconsistent.*

The proof of theorem 3 constitutes part III. It is assumed that

$$D = 0, \text{ and at least one of } D_1, D_2, D_3 \text{ is not zero.} \tag{57}$$

The theorem will be proved by showing that, if there is a solution of (26) and if conditions (57) hold, then there is a contradiction. By part I equations (51) are true. If also $D = 0$, then $D_1 = 0$, $D_2 = 0$, $D_3 = 0$. This contradicts the last part of (57).

PROBLEMS

Use theorems 2 and 3 to show that the equations in each of the following systems are inconsistent or have a unique solution. If the solution is unique determine it

1. $$\begin{aligned} 3x - y + 7z &= 2 \\ x + y - 2z &= 1 \\ 5x + 3y + 4z &= -3 \end{aligned}$$

2. $$\begin{aligned} v + 2s + t &= 2, \\ 2v + 5s - 2t &= -1 \\ -3v - 7s + 4t &= 2 \end{aligned}$$

3. $$\begin{aligned} 3x + 2y - z &= -7 \\ x + y - z &= -2 \\ x - 3y + 7z &= 2 \end{aligned}$$

4. $$\begin{aligned} u - 5v + 3w &= 0 \\ -2u + 3v + w &= 7 \\ 3u - 7v + w &= 1 \end{aligned}$$

5. $$\begin{aligned} 2v - s + t &= 0 \\ 7v + 3s - t &= 0 \\ v + 5s + 2t &= 0 \end{aligned}$$

6. $$\begin{aligned} x + 2y - z &= 3 \\ 2x + 3y + z &= -1 \\ -4x - 5y - 5z &= 2 \end{aligned}$$

7. $$\begin{aligned} x + 19y + 2z &= 1 \\ 2x \qquad y - z &= 9 \\ 9x - 24y - 7z &= 13 \end{aligned}$$

8. $$\begin{aligned} 2u + v - 5w &= 0 \\ u + 7v - w &= 0 \\ 4u + 3v + w &= 0 \end{aligned}$$

9. $$\begin{aligned} 2u + v + w &= 6 \\ 3u + 5v - w &= -2 \\ -u + v + w &= 0 \end{aligned}$$

10. $$\begin{aligned} x - y - 7z &= 2 \\ -4x - y + 3z &= 5 \\ 7x + 3y + z &= 1 \end{aligned}$$

11. $$\begin{aligned} v + s + t &= 1 \\ 2v - s + 3t &= 7 \\ 9v - 6s + 14t &= 2 \end{aligned}$$

12. $$\begin{aligned} 2x + 3y - z &= 6 \\ x - 7y + 3z &= -9 \\ 5x + y + 2z &= 4 \end{aligned}$$

Part IV is devoted to a statement without proof of facts for three linear equations in three unknowns which are a special case of similar facts for an arbitrary number n of unknowns and an arbitrary number q of equations. These facts will be proved in chapter 7. In part IV it is assumed that

$$(58) \qquad D = 0, \quad D_1 = 0 \quad D_2 = 0 \quad D_3 = 0$$

Thus the results to be stated in part IV and the results in theorem 3 complete the discussion of the case $D = 0$ for equations (26). Since theorem 1 and theorem 2 completed the discussion of the case $D \neq 0$ all the possibilities for equations (26) will have been considered

If the hypothesis of part I is used (51) are necessary conditions. If also (58) are assumed and used in (51) the true statement

$$(59) \qquad 0\, x_1 = 0 \quad 0\, x_2 = 0 \quad 0\, x_3 = 0$$

results. Since (59) gives no information about x_1, x_2, x_3, part I is not used in part IV. In part IV equations (26) are considered

with x_1, x_2, x_3 as variables. This is the way in which (26) were interpreted in the statements of theorems 1, 2, and 3. It was only in part I, and in those proofs in parts II and III which used results of part I, that x_1, x_2, x_3 were considered temporarily as constants, as explained at the beginning of part I.

A set (60) of three numerical equations will be given which satisfy (58) and which have infinitely many solutions. Then a set (67) of three numerical equations will be given which satisfy (58) but which have no solution. Hence it will be clear that new methods are needed to complete the discussion of the set (26) of equations for which (58) are true. These new methods will be illustrated by means of systems (60) and (67).

If (40), (42), (48) are used, it is found that

$$\begin{aligned} x + 2y - 3z &= 2, \\ 2x - y + z &= -3, \\ 4x - 7y + 9z &= -13 \end{aligned} \tag{60}$$

satisfy (58). Next it will be proved that each solution of the first two of these equations is a solution of the third equation. It is obvious that

$$4x - 7y + 9z + 13 \equiv -2(x + 2y - 3z - 2) + 3(2x - y + z + 3). \tag{61}$$

Therefore (61) is true for all values of x, y, z. Therefore a particular set of values of x, y, z for which each quantity enclosed by parentheses on the right-hand side of (61) is zero is a set of values for which the left-hand side is zero. Hence a solution of the first two of equations (60) is a solution of the last of these equations. Now by (3) and (7) it is true that the first two of equations (60) have infinitely many solutions. Hence the system (60) is a system with infinitely many solutions.

The proof of these facts about the solution of (60) is simple because the number n of variables is small. If n were greater than three, or if the number q of equations were greater than three, not only the proof but indeed the statement of the facts might be complicated. These statements and proof are simplified by the use of the ideas of matrix and rank of a matrix. These ideas will now be illustrated with equations (60). The coefficients of the variables and the constants on the right-hand sides of the equa-

tions in (60) when written down in the order in which they occur in (60) form an array

$$(62)\qquad \begin{bmatrix} 1 & 2 & -3 & 2 \\ 2 & -1 & 1 & -3 \\ 4 & -7 & 9 & -13 \end{bmatrix}$$

This array is called a *matrix* A matrix is not a determinant because a matrix is merely a rectangular (perhaps square) array of numbers In (62) the smaller array

$$(63)\qquad \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 4 & -7 & 9 \end{bmatrix}$$

is formed by he coefficients of the variables The matrix (63) is called *the coefficient matrix* of equations (60) The notation c m will be used to designate the matrix (63) The matrix (62) is called *the augmented matrix* of the equations (60) It will be designated by the notation a m

The matrix (63) suggests the determinant D in (40) Indeed any square matrix suggests the determinant whose elements are respectively the elements of the matrix The matrix is merely the ordered array of numbers Its determinant is a number The a m (62) suggests four determinants of order three These are the determinant D and

$$(64)\qquad \begin{vmatrix} 1 & 2 & 2 \\ 2 & -1 & -3 \\ 4 & -7 & -13 \end{vmatrix} \quad \begin{vmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & 9 & -13 \end{vmatrix} \quad \begin{vmatrix} 2 & -3 & 2 \\ -1 & 1 & -3 \\ -7 & 9 & -13 \end{vmatrix}$$

The first determinant in (64) is D_3 for (60) by the last part of (48) The second determinant in (64) is not quite D_2 and the third determinant in (64) is not quite D_1 because by (48_1) and (42)

$$(65)\qquad D_2 = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ 4 & -13 & 9 \end{vmatrix} \quad D_1 = \begin{vmatrix} 2 & 2 & -3 \\ -3 & -1 & 1 \\ -13 & -7 & 9 \end{vmatrix}$$

Also it can be verified by (54) that the third determinant in (64) equals D_1 in (65) and that the second determinant in (64) equals $-D_2$ in (65) Hence the fact that (60) satisfy (58) is equivalent to the statement that each determinant of order three which can be formed from the a m is zero

In (62) there are many submatrices which have two rows and two columns. For example, the first two rows of (62) yield the following submatrices:

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix},$$
$$\begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -1 & -3 \end{bmatrix}, \begin{bmatrix} -3 & 2 \\ 1 & -3 \end{bmatrix}.$$

Any matrix which has two rows and two columns suggests a determinant of order two. For example, the first matrix above suggests

$$(66) \qquad \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}.$$

By (52) and (53) this is the number -5. Therefore in (62) each of the determinants of order three is zero, and at least one of the determinants of the order two is not zero. This is the meaning of the statement that the rank of (62) is two. In general, *the rank of a matrix* is the number of rows in the largest non-zero determinant which can be formed from the matrix. The rank of the a.m. will be designated by r_a.

The coefficient matrix (63) has a rank. This rank will be designated by r. It is especially to be noted that the c.m. is a submatrix of the a.m. Hence a non-zero determinant which can be formed from (63) is a non-zero determinant which can be formed from (62). Thus $r \leqq r_a = 2$. Also, (66) is a non-zero determinant of the second order which can be formed from (63). Therefore $r = 2$, and $r_a = 2$. It will be proved later for an arbitrary number n of variables that, *if $r_a = r$, then the equations have a solution and that, if also $r < n$, then the equations have infinitely many solutions.* Equations (60) illustrate these facts, since $r = 2$, $r_a = 2$, $n = 3$.

These facts which equations (60) illustrate are to be contrasted with the facts which the equations

$$(67) \qquad \begin{aligned} x + 2y - 3z &= 2, \\ -3x - 6y + 9z &= 7, \\ 2x + 4y - 6z &= 5, \end{aligned}$$

illustrate. An *s-rowed minor* of a matrix is, by definition, a deter-

minant of order s which can be formed from the matrix All the three-rowed minors of the a m

$$(68)\qquad \begin{bmatrix} 1 & 2 & -3 & 2 \\ -3 & -6 & 9 & 7 \\ 2 & 4 & -6 & 5 \end{bmatrix}$$

are zero Hence (67) satisfy (58) All two-rowed minors of the c m

$$(69)\qquad \begin{bmatrix} 1 & 2 & -3 \\ -3 & -6 & 9 \\ 2 & 4 & -6 \end{bmatrix}$$

are zero Therefore $r = 1$ There is at least one non-zero two-rowed minor of the a m, namely, the determinant formed by the four elements in the upper right-hand corner of (68) Therefore $r_a = 2$ It will be proved later for arbitrary n and arbitrary q that, *if $r < r_a$, then the equations have no solution* This fact is illustrated by (67) because the discussion of equations (8) showed that the first two equations of (67) have no solution and hence the system (67) has no solution

The solution of equations (9) can be discussed by means of ranks For these equations $r = 3$, $r_a = 3$, $n = 3$ Thus these equations illustrate the fact that $r_a = r$ The equations have a unique solution, by (12) Therefore the solution of (9) illustrates the theorem which will be proved for arbitrary n and arbitrary q that, *if $r_a = r$ and if $r = n$ then there is one and only one solution* Equations (9) do not satisfy (58), and they do not satisfy (57)

PROBLEMS

Find the ranks r and r_a for each of the following systems of equations Use the facts concerning ranks which have been stated in part IV to determine whether the equations in each system are inconsistent or consistent If they are consistent determine by ranks whether there are infinitely many solutions or only one solution

1 $5x + 3y + z = 18,\quad -7x - 2y + 2z = 10,\quad 9x + 2y - z = 5$

2 $x - y + z = 13,\quad 5x + 2y - 2z = 7,\quad -2x - 5y + 5z = 32$

3 $u + 2s + t = 2,\quad 5u + s - t = -3,\quad -13u + 4s + 7t = -11$

4 $2u - s + t = -5,\quad -3u + 2s + 2t = 3,\quad 7u + 5s + 9t = 14$

5. $$\begin{aligned} u - v - 2w &= 0,\\ 2u + 5v - 3w &= 0,\\ 3u - 17v - 8w &= 0. \end{aligned}$$

6. $$\begin{aligned} 3u - v + w &= 1,\\ 2u + 4v - w &= 7,\\ 5u + 17v - 5w &= 11. \end{aligned}$$

7. $$\begin{aligned} x - 2y + z &= 1,\\ 3x - 6y + 3z &= 2,\\ -2x + 4y - 2z &= 11. \end{aligned}$$

8. $$\begin{aligned} 2x + 3y - 2z &= 0,\\ x + 7y - 5z &= 0,\\ 4x + 2y + z &= 0. \end{aligned}$$

9. $$\begin{aligned} 7x + y + z &= 8,\\ 2x - 3y - z &= 1,\\ 4x + 17y + 7z &= 11. \end{aligned}$$

10. $$\begin{aligned} 3u + v - 2w &= 4,\\ 6u + 2v - 4w &= 7,\\ -3u - v + 2w &= -4. \end{aligned}$$

11. $$\begin{aligned} 7u + 4s - t &= 1,\\ 2u - s + 2t &= 5,\\ -3u + 2s + 5t &= 17. \end{aligned}$$

12. $$\begin{aligned} u - 5s + 2t &= 3,\\ 4u - s + t &= 1,\\ -8u - 17s + 5t &= 7. \end{aligned}$$

CHAPTER 6

DETERMINANTS

1 Determinants of order four Determinants of order four are numbers which occur in the solution of four linear equations in four unknowns This will be explained by means of the equations

$$
(1)\qquad
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= k_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= k_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= k_3 \\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= k_4
\end{aligned}
$$

It is specifically assumed that there actually are four variables in these equations and that there actually are four equations in the set The methods of section 3 of chapter 5 which led to equations (51) are applicable here Thus for example first x_4 could be eliminated between (1_1) and (1_2) next x_4 would be eliminated between (1_1) and (1_3) then x_4 would be eliminated between (1_1) and (1_4) There would result three equations in x_1 x_2 x_3 Then these three equations would be treated as equations (26) of chapter 5 were treated In whatever way the eliminations were performed there would result four necessary conditions analogous to the three necessary conditions (51) of chapter 5 In each of these four conditions the coefficient of the variable is the number

$$
(2)\qquad
\begin{aligned}
&+ a_{11}a_{22}a_{33}a_{44} + a_{21}a_{32}a_{13}a_{44} + a_{31}a_{12}a_{23}a_{44} - a_{41}a_{12}a_{23}a_{34} \\
&- a_{11}a_{32}a_{23}a_{44} - a_{21}a_{12}a_{33}a_{44} - a_{31}a_{22}a_{13}a_{44} + a_{41}a_{12}a_{33}a_{24} \\
&- a_{11}a_{22}a_{43}a_{34} - a_{21}a_{32}a_{43}a_{14} - a_{31}a_{12}a_{43}a_{24} - a_{41}a_{22}a_{33}a_{14} \\
&+ a_{11}a_{32}a_{43}a_{24} + a_{21}a_{12}a_{43}a_{34} + a_{31}a_{22}a_{43}a_{14} + a_{41}a_{22}a_{13}a_{34} \\
&+ a_{11}a_{42}a_{23}a_{34} + a_{21}a_{42}a_{33}a_{14} + a_{31}a_{42}a_{13}a_{24} - a_{41}a_{32}a_{13}a_{24} \\
&- a_{11}a_{42}a_{33}a_{24} - a_{21}a_{42}a_{13}a_{34} - a_{31}a_{42}a_{23}a_{14} + a_{41}a_{32}a_{23}a_{14}
\end{aligned}
$$

This number will be designated by D. It is *the determinant of order four whose symbol is*

$$\text{(3)}\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

A rule will now be explained by which the number (2) can be written down directly from the symbol (3). All the arrangements of the numbers 1, 2, 3, 4 are given in the first column of Table I. These arrangements can be found by writing the six arrangements of 1, 2, 3 and then inserting the number 4 in all possible positions.

TABLE I

Arrangement	p	$(-1)^p$	Literal product	Signed product
1 2 3 4	0	1	$a_{11}a_{22}a_{33}a_{44}$	$+\,a_{11}a_{22}a_{33}a_{44}$
1 3 2 4	1	−1	$a_{11}a_{32}a_{23}a_{44}$	$-\,a_{11}a_{32}a_{23}a_{44}$
2 3 1 4	2	1	$a_{21}a_{32}a_{13}a_{44}$	$+\,a_{21}a_{32}a_{13}a_{44}$
2 1 3 4	1	−1	$a_{21}a_{12}a_{33}a_{44}$	$-\,a_{21}a_{12}a_{33}a_{44}$
3 1 2 4	2	1	$a_{31}a_{12}a_{23}a_{44}$	$+\,a_{31}a_{12}a_{23}a_{44}$
3 2 1 4	3	−1	$a_{31}a_{22}a_{13}a_{44}$	$-\,a_{31}a_{22}a_{13}a_{44}$
1 2 4 3	1	−1	$a_{11}a_{22}a_{43}a_{34}$	$-\,a_{11}a_{22}a_{43}a_{34}$
1 3 4 2	2	1	$a_{11}a_{32}a_{43}a_{24}$	$+\,a_{11}a_{32}a_{43}a_{24}$
2 3 4 1	3	−1	$a_{21}a_{32}a_{43}a_{14}$	$-\,a_{21}a_{32}a_{43}a_{14}$
2 1 4 3	2	1	$a_{21}a_{12}a_{43}a_{34}$	$+\,a_{21}a_{12}a_{43}a_{34}$
3 1 4 2	3	−1	$a_{31}a_{12}a_{43}a_{24}$	$-\,a_{31}a_{12}a_{43}a_{24}$
3 2 4 1	4	1	$a_{31}a_{22}a_{43}a_{14}$	$+\,a_{31}a_{22}a_{43}a_{14}$
1 4 2 3	2	1	$a_{11}a_{42}a_{23}a_{34}$	$+\,a_{11}a_{42}a_{23}a_{34}$
1 4 3 2	3	−1	$a_{11}a_{42}a_{33}a_{24}$	$-\,a_{11}a_{42}a_{33}a_{24}$
2 4 3 1	4	1	$a_{21}a_{42}a_{33}a_{14}$	$+\,a_{21}a_{42}a_{33}a_{14}$
2 4 1 3	3	−1	$a_{21}a_{42}a_{13}a_{34}$	$-\,a_{21}a_{42}a_{13}a_{34}$
3 4 1 2	4	1	$a_{31}a_{42}a_{13}a_{24}$	$+\,a_{31}a_{42}a_{13}a_{24}$
3 4 2 1	5	−1	$a_{31}a_{42}a_{23}a_{14}$	$-\,a_{31}a_{42}a_{23}a_{14}$
4 1 2 3	3	−1	$a_{41}a_{12}a_{23}a_{34}$	$-\,a_{41}a_{12}a_{23}a_{34}$
4 1 3 2	4	1	$a_{41}a_{12}a_{33}a_{24}$	$+\,a_{41}a_{12}a_{33}a_{24}$
4 2 3 1	5	−1	$a_{41}a_{22}a_{33}a_{14}$	$-\,a_{41}a_{22}a_{33}a_{14}$
4 2 1 3	4	1	$a_{41}a_{22}a_{13}a_{34}$	$+\,a_{41}a_{22}a_{13}a_{34}$
4 3 1 2	5	−1	$a_{41}a_{32}a_{13}a_{24}$	$-\,a_{41}a_{32}a_{13}a_{24}$
4 3 2 1	6	1	$a_{41}a_{32}a_{23}a_{14}$	$+\,a_{41}a_{32}a_{23}a_{14}$

In each row of the second column of this table is the number p of inversions in the arrangement which is in that row. In the

third column are the values of $(-1)^p$ In each row of the fourth column is the literal product whose factors have their second subscripts *in the normal order and their first subscripts in the arrangement* appearing in that row In each row of the fifth column is the signed product, which is the result of multiplying the literal product and $(-1)^p$ for that row Now, by definition, the determinant of the fourth order whose symbol is (3) is the number (2), that is, the sum of all the 4! signed products in column five of the table

Next there will be explained a notation which is used to describe in one phrase all the signed products which occur in (2) The arrangement which the first subscripts of a signed product form is designated by $i_1i_2i_3i_4$ For the particular signed product $-a_{31}a_{12}a_{43}a_{24}$, therefore $i_1 = 3$ $i_2 = 1$, $i_3 = 4$, $i_4 = 2$ Also for arrangement 3142 of its first subscripts the table gives $p = 3$ Hence this signed product is a special instance of *the arbitrary signed product* $(-1)^p a_{i_11}a_{i_22}a_{i_33}a_{i_44}$, *in which* $i_1i_2i_3i_4$ *is an arrangement of the numbers 1, 2, 3, 4, showing* p *inversions* Again, the signed product $+a_{21}a_{42}a_{33}a_{14}$ is the special instance of the arbitrary signed product in which $i_1 = 2$, $i_2 = 4$, $i_3 = 3$, $i_4 = 1$, and $p = 4$ Each of the 4! signed products which occur in (2) is an instance of the arbitrary signed product designated above All the signed products of this type occur in (2) Therefore the determinant (2) is the sum of all the terms of this type This is a rule by which (2) may be written down directly from (3)

Another rule by which the number (2) may be written down directly from the symbol (3) will be explained now There are six terms in (2) which involve the factor a_{11} The sum of these six terms is $a_{11}(a_{22}a_{33}a_{44} - a_{32}a_{23}a_{44} - a_{22}a_{43}a_{34} + a_{32}a_{43}a_{24} + a_{42}a_{23}a_{34} - a_{42}a_{33}a_{24})$ The coefficient of a_{11} in this expression can be written in the form $a_{22}(a_{33}a_{44} - a_{43}a_{34}) - a_{32}(a_{23}a_{44} - a_{43}a_{24}) + a_{42}(a_{23}a_{34} - a_{33}a_{24})$ In this form it is obvious, by analogy with (54) of chapter 5, that this coefficient of a_{11} is the determinant

$$\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

Now this symbol is obtained from the symbol (3) by deleting from the symbol (3) *the row and column in which* a_{11} *stands* This determinant is called *the minor of* a_{11}, and is designated by A_{11}

Hence the sum of the terms in (2) which involve a_{11} is $a_{11}A_{11}$. The minor

$$\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

of a_{21} in (3) is designated by A_{21}. Analogous definitions hold for A_{31} and A_{41}. With these notations the number (2) becomes $a_{11}A_{11} - a_{21}A_{21} + a_{31}A_{31} - a_{41}A_{41}$. This expression gives a practical rule for writing down the number (2) directly from the symbol (3). Other facts about determinants of order four will appear as special cases of rules for determinants of order n.

The determinant whose symbol is

$$(4) \qquad \begin{vmatrix} k_1 & a_{12} & a_{13} & a_{14} \\ k_2 & a_{22} & a_{23} & a_{24} \\ k_3 & a_{32} & a_{33} & a_{34} \\ k_4 & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

will be designated by D_1. Thus D_1 is a number which can be obtained from (2) by replacing a_{11}, a_{21}, a_{31}, a_{41} respectively by k_1, k_2, k_3, k_4. Similarly D_2 is, by definition, the determinant whose symbol is obtained from the symbol (3) of D by replacing the elements in the second column of (3) respectively by the constants k_1, k_2, k_3, k_4. Also D_3 is the determinant whose symbol is obtained from the symbol (3) by replacing the third column of (3) by k_1, k_2, k_3, k_4, and D_4 is obtained by replacing the fourth column of (3) by k_1, k_2, k_3, k_4. Then the four necessary conditions which were mentioned just before equation (2) can be written

$$(5) \qquad Dx_1 = D_1, \quad Dx_2 = D_2, \quad Dx_3 = D_3, \quad Dx_4 = D_4.$$

The proofs of the following fundamental theorems, in which $n = 4$, can be completed as the analogous proofs in chapter 5 were completed.

THEOREM 1. *If the determinant D of the coefficient matrix of the system (1) is not zero, then there is one and only one solution. This solution is the ordered set of numbers D_1/D, D_2/D, D_3/D, D_4/D.*

THEOREM 2. *If the determinant D of the coefficient matrix of the system (1) is zero, and if at least one of the determinants D_1, D_2, D_3, D_4 is not zero, then the equations are inconsistent.*

PROBLEMS

Find the ranks r and r_a for each of the following systems of equations. Use theorems 1 and 2 to determine whether the equations are consistent or inconsistent. If they are consistent, solve them.

1
$$\begin{aligned} 3x + 7y + z - 2w &= -10 \\ -x + 2y + 5z + w &= 6 \\ 2x + 6y + 4z - w &= -3 \\ -5x + 2y + 9z + w &= 2 \end{aligned}$$

2
$$\begin{aligned} 2v + 5s - t + u &= 4 \\ -v + s + 7t - u &= 11 \\ v + 3s - 6t + 11u &= 8 \\ 9v + 13s + t + u &= 1 \end{aligned}$$

3
$$\begin{aligned} v + 2s - t + 5 &= 1 \\ 4v - s + 3t + u &= 7 \\ -v - 3s + 5t - u &= 3 \\ -5v - 3s + 6t + 2u &= 2 \end{aligned}$$

4.
$$\begin{aligned} x - y + z + 5w &= 1 \\ 3x + 2y - 2z + w &= 2 \\ -x + 3y - 2z - 7w &= 4 \\ -4x + 5y - 2z - 12w &= 3 \end{aligned}$$

5
$$\begin{aligned} 2x - 3y + 7z + w &= 0 \\ -x + 2y + 5z &= 0 \\ 3x + 4y + z + w &= 0 \\ 2x - 5y + 3z + 4w &= 0 \end{aligned}$$

6
$$\begin{aligned} 7v + 2s - t + u &= 0 \\ 2v - 3s + 5t + 4u &= 0 \\ v + 11t + 2u &= 0 \\ -2v + s + 9t + u &= 0 \end{aligned}$$

7
$$\begin{aligned} x - 2y + z + 3w &= 2 \\ 2x - y + 5z - w &= 1 \\ 2x + 7y + 3z + w &= -1 \\ 3x - 14y + 5z + 7w &= 0 \end{aligned}$$

8
$$\begin{aligned} 3x + y - 2z + 11w &= -12 \\ -x + 7y + z - w &= 2 \\ 2x + 2y + 5z &= 12 \\ x - y + 4z + 2w &= 7 \end{aligned}$$

9
$$\begin{aligned} 2v - s \quad t - w &= 2 \\ v + 5s + 2t + w &= -9 \\ 3v + 2s - 7t - 5w &= -6 \\ -v + 3s + 5t + 2w &= -3 \end{aligned}$$

10
$$\begin{aligned} u - v - w + t &= 1 \\ 2u + 3v + w - 3t &= 2 \\ 5u - v - 3w + 2t &= -1 \\ -15u - 2v + 6w - t &= 1 \end{aligned}$$

It is to be noted that in this section there has been a discussion only of the case in which $n = 4$ and $q = 4$. Also if each of D, D_1, D_2, D_3, D_4 is zero, then theorems 1 and 2 are not applicable. That is if $r < n$ and $r_a < n$, then further discussion is required. No illustrations will be given here which are analogous to the systems (60) and (67) of chapter 5. All the situations which may arise will appear as special cases after the general theorems have been proved for arbitrary n and arbitrary q.

2 Determinants of order five. Determinants of order n. If n is a positive integer, then the notation

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = k_1$$

(6) $$\cdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = k_n$$

will be used for a system of n linear equations in n unknowns. The particular case in which $n = 3$ was discussed in chapter 5.

The results were simplified by defining determinants of order three. The particular case in which $n = 4$ was discussed in section 1, and determinants of order four were defined in that discussion. If $n = 5$, analogous details would be very intricate. They will not be presented here. The results will be obtained very simply as a special case after the general theorems have been proved for arbitrary n. However, the definition of a determinant of the fifth order will be given here to illustrate the fundamental definition of a determinant of order n.

There are 5! arrangements of the numbers 1, 2, 3, 4, 5. These arrangements may be obtained systematically in the following manner from the 4! arrangements of the numbers 1, 2, 3, 4 which are listed in the first column of Table I in section 1. First adjoin the number 5 on the right of each of the arrangements in Table I. Then insert 5 between the last two numbers of each of the arrangements in Table I. Then insert 5 between the second and third numbers in each arrangement. Then insert 5 between the first and second numbers in each arrangement. Then adjoin 5 on the left of each of the arrangements. Thus the table of arrangements of 1, 2, 3, 4, 5 would have five sections, each section derived from the first column of Table I. A portion of one of these sections is given in the first column of Table II.

TABLE II

Arrangement	p	$(-1)^p$	Literal product	Signed product
1 5 2 4 3	4	1	$a_{11}a_{52}a_{23}a_{44}a_{35}$	$+\,a_{11}a_{52}a_{23}a_{44}a_{35}$
1 5 3 4 2	5	-1	$a_{11}a_{52}a_{33}a_{44}a_{25}$	$-\,a_{11}a_{52}a_{33}a_{44}a_{25}$
2 5 3 4 1	6	1	$a_{21}a_{52}a_{33}a_{44}a_{15}$	$+\,a_{21}a_{52}a_{33}a_{44}a_{15}$
2 5 1 4 3	5	-1	$a_{21}a_{52}a_{13}a_{44}a_{35}$	$-\,a_{21}a_{52}a_{13}a_{44}a_{35}$
3 5 1 4 2	6	1	$a_{31}a_{52}a_{13}a_{44}a_{25}$	$+\,a_{31}a_{52}a_{13}a_{44}a_{25}$
3 5 2 4 1	7	-1	$a_{31}a_{52}a_{23}a_{44}a_{15}$	$-\,a_{31}a_{52}a_{23}a_{44}a_{15}$

If the entire table for $n = 5$ were exhibited, there would be 5! signed products in the last column. The sum of these 5! signed products is, by definition, the determinant whose symbol is

$$(7) \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}.$$

The notation $(-1)^p a_{i_11}a_{i_22}a_{i_33}a_{i_44}a_{i_55}$ is used to describe all the signed products whose sum is the determinant whose symbol is (7) The particular signed product $-a_{21}a_{52}a_{13}a_{44}a_{35}$ has $p = 5$, $i_1 = 2$, $i_2 = 5$, $i_3 = 1$, $i_4 = 4$, $i_5 = 3$ Again, the signed product $+a_{31}a_{52}a_{13}a_{44}a_{25}$ has $p = 6$ $i_1 = 3$, $i_2 = 5$, $i_3 = 1$, $i_4 = 4$, $i_5 = 2$ Hence, by definition the determinant of order five whose symbol is (7) is the sum of the $5!$ terms of the type $(-1)^p a_{i_11}a_{i_22}a_{i_33}a_{i_44}a_{i_55}$ in which $i_1i_2i_3i_4i_5$ is an arrangement of the numbers 1 2 3 4, 5 showing p inversions

In general by definition the determinant of order n whose symbol is

$$\text{(8)} \qquad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

is the sum of the $n!$ terms of the type $(-1)^p a_{i_11}a_{i_22} \cdots a_{i_nn}$ in which $i_1i_2 \cdots i_n$ is an arrangement of the numbers $1, 2, \cdots, n$ showing p inversions

A practical rule for writing down the number which is the determinant directly from the symbol of the determinant was proved if $n = 3$ and if $n = 4$ This was the expansion of the determinant by minors of the elements of its first column The proofs of this rule and other expansion rules for determinants of order n involve the use of two fundamental properties of determinants These fundamental properties are also used to prove facts which simplify calculations with determinants The simple proofs, which will be given later of the facts about an arbitrary number q of linear equations in an arbitrary number n of unknowns are based directly on these properties of determinants of order n

3 First and second fundamental properties of determinants of order n The idea of one-to-one correspondence is basic in the proofs of these properties and in many other mathematical proofs The form in which it is to be used will now be illustrated Let there be a set of seven numbers s_1 s_2, $\cdots$, s_7, and let S be their sum Let there be a second set of seven numbers, t_1, t_2 $\cdots$, t_7 and let T be their sum Therefore $S = s_1 + s_2 + \cdots + s_7$, and $T = t_1 + t_2 + \cdots + t_7$ Now if it were known that $s_1 = t_1$,

$s_2 = t_2, \cdots, s_7 = t_7$, then it would be true that $S = T$. These seven equations, which are the hypothesis that implies $S = T$, are an illustration of the meaning of the statement that the seven numbers in the first set and the seven numbers in the second set have been paired and that the numbers in each pair are equal. In general, a pairing, regardless of whether the numbers in each pair are equal or not, is called a one-to-one correspondence. Another one-to-one correspondence in which corresponding numbers are equal is illustrated by the equations $s_1 = t_7$, $s_2 = t_3$, $s_3 = t_2$, $s_4 = t_1$, $s_5 = t_6$, $s_6 = t_5$, $s_7 = t_4$. If this were the hypothesis, it would be true that $S = T$. A one-to-one correspondence in which corresponding numbers are negatives of each other is illustrated by the equations $s_1 = -t_2$, $s_2 = -t_5$, $s_3 = -t_3$, $s_4 = -t_1$, $s_5 = -t_7$, $s_6 = -t_6$, $s_7 = -t_4$. If this were the hypothesis, it would be true that $S = -T$. It is obvious that the number seven of summands could be replaced by any positive integer. Hence the following lemma has been proved.

LEMMA 1. *Let N be a positive integer, and let there be two sets of numbers with N numbers in each set. If a one-to-one correspondence exists between the numbers in the two sets such that corresponding numbers are equal, then the sum of the numbers in the first set equals the sum of the numbers in the second set. If a one-to-one correspondence exists such that corresponding numbers are negatives of each other, then the sum of the numbers in the first set is the negative of the sum of the numbers in the second set.*

The following lemma 2 is also basic in the proofs of the first and second fundamental properties of determinants of order n. This lemma will be illustrated now for the case that $n = 7$. From the arrangement 3 5 1 7 4 2 6 of the numbers 1, 2, 3, 4, 5, 6, 7 obtain, by interchanging the numbers 5 and 2, the arrangement 3 2 1 7 4 5 6. There are nine inversions in the first arrangement and six inversions in the last arrangement. The difference between nine inversions and six inversions is the odd number three. This illustrates lemma 2, because lemma 2 states that the difference in the numbers of inversions is an odd integer. The proof of lemma 2 is illustrated in the following explanation of how the number of inversions changes from nine to six. Let the two adjacent numbers 5 and 1 in the arrangement 3 5 1 7 4 2 6 be interchanged. All the inversions which are in the original inversion,

except that due to 5 and 1 appear also in the new arrangement 3 1 5 7 4 2 6 No new inversions can appear In this second arrangement let the two adjacent numbers 5 and 7 be interchanged The third arrangement is 3 1 7 5 4 2 6 All the inversions which were in the second arrangement also appear in the third arrangement and one new inversion due to 7 and 5 appears in the third arrangement This illustrates the general fact that if adjacent numbers are interchanged in an arrangement then the number of inversions in the new arrangement is one more or one less than the number of inversions in the original arrangement

Now continue to interchange 5 with each number on its right until it has been interchanged finally with the number 2 Then interchange 2 with each number on its left until it has been interchanged with the number 1 The following table exhibits the arrangements and shows how the number of inversions changes

Arrangement	Number of inversions
3 5 1 7 4 2 6	9
3 1 5 7 4 2 6	8
3 1 7 5 4 2 6	9
3 1 7 4 5 2 6	8
3 1 7 4 2 5 6	7
3 1 7 2 4 5 6	6
3 1 2 7 4 5 6	5
3 2 1 7 4 5 6	6

The proof of lemma 2 involves only the ideas illustrated in the preceding discussion Let n be a positive integer If two arrangements of the numbers 1 2 3 n are such that one interchange of adjacent numbers in the first arrangement yields the second arrangement then the number of inversions in the first arrangement minus the number of inversions in the second arrangement is 1 or -1 Next consider two arrangements such that one interchange of non adjacent numbers in the first arrangement yields the other arrangement Let there be t numbers in the first arrangement which appear between the two numbers to be interchanged Then there are $2t + 1$ arrangements which can be tabulated under the first of the two given arrangements such that each arrangement *is derived from the preceding one by interchanging adjacent* numbers and such that the last arrangement in the tabulation is the other of the two given arrangements Thus the number of

inversions in the first arrangement is obtained from the number of inversions in the last arrangement by adding $2t + 1$ integers, each of which is 1 or -1. This completes the proof of lemma 2.

LEMMA 2. *If two arrangements of the numbers 1, 2, 3, $\cdots$, n are so related that one interchange of two numbers in the first arrangement gives the second arrangement, then the number of inversions in the first arrangement is the sum of the number of inversions in the second arrangement and an odd (positive or negative) integer.*

Consider two determinants A and B whose symbols are respectively

$$(9)\qquad \begin{vmatrix} a_{11} & \cdots & a_{14} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{41} & \cdots & a_{44} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} b_{11} & \cdots & b_{14} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_{41} & \cdots & b_{44} \end{vmatrix}.$$

By definition, A is the sum of the signed products in the last column of Table I. This table will temporarily be referred to as Table I_a. The table which is obtained from Table I_a by replacing each letter a by the letter b will be referred to as Table I_b. Then, by definition, B is the sum of the signed products of the last column of Table I_b. It is to be noted especially that all subscripts in Table I_a remain precisely as they are when Table I_b is formed and that the signs of the signed products remain. For example, the signed product $+ a_{21}a_{32}a_{13}a_{44}$ in Table I_a is in the same location as the signed product $+ b_{21}b_{32}b_{13}b_{44}$ in Table I_b; the signed product $- a_{21}a_{42}a_{13}a_{34}$ in the same location as $- b_{21}b_{42}b_{13}b_{34}$.

The first fundamental property, which will be illustrated and proved now if $n = 4$, has the following hypothesis:

first	column of symbol of A is first	column of symbol of B;
second	column of symbol of A is third	column of symbol of B;
third	column of symbol of A is second	column of symbol of B;
fourth	column of symbol of A is fourth	column of symbol of B.

This hypothesis is also expressed by the statement that A is obtained by interchanging the second and third columns of B. In terms of the elements of the symbols (9) this hypothesis is

$$(10)\qquad a_{i1} = b_{i1},\quad a_{i2} = b_{i3},\quad a_{i3} = b_{i2},\quad a_{i4} = b_{i4}\quad (i = 1, 2, 3, 4).$$

The conclusion in the first fundamental property is that $A = -B$ This will be proved by means of lemmas 1 and 2 and equations (10)

It will be proved that there is a one-to-one correspondence between the signed products in the last column of Table I_a and the signed products in the last column of Table I_b such that corresponding signed products are negatives of each other For example consider the signed product $+ a_{11}a_{22}a_{33}a_{44}$ in I_a By (10) it is true that $+ a_{11}a_{22}a_{33}a_{44} = + b_{11}b_{23}b_{32}b_{44}$ Now any product is the same regardless of the order in which the factors are written down since the factors are ordinary numbers Hence $+ b_{11}b_{23}b_{32}b_{44} = + b_{11}b_{32}b_{23}b_{44}$ Hence $+ a_{11}a_{22}a_{33}a_{44} = + b_{11}b_{32}b_{23}b_{44}$ Since the left-hand side of this equality is an entry in column five of Table I_a therefore the right hand side namely $+ b_{11}b_{32}b_{23}b_{44}$ is a term in the sum which is A although it does not look like a term in A Indeed $+ b_{11}b_{32}b_{23}b_{44}$ looks more like the terms in B because terms in B are signed products of four factors each of which is a double-subscripted letter b But $- b_{11}b_{32}b_{23}b_{44}$ (not $+ b_{11}b_{32}b_{23}b_{44}$) is in the last column of Table I_b This is true because the literal product $b_{11}b_{32}b_{23}b_{44}$ is found in Table I_b precisely where the literal product $a_{11}a_{32}a_{23}a_{44}$ is found in Table I_a namely in the second row Therefore in Table I_b the signed product $- b_{11}b_{32}b_{23}b_{44}$ is in the second row and last column Thus it has been proved that the signed product $+ a_{11}a_{22}a_{33}a_{44}$ in A equals the negative of the signed product $- b_{11}b_{32}b_{23}b_{44}$ in B

Next consider the signed product $- a_{11}a_{32}a_{23}a_{44}$ of Table I_a By (10) it is true that $- a_{11}a_{32}a_{23}a_{44} = - b_{11}b_{33}b_{22}b_{44}$ By rearranging factors it is true that $- b_{11}b_{33}b_{22}b_{44} = - b_{11}b_{22}b_{33}b_{44}$ Hence $- a_{11}a_{32}a_{23}a_{44} = - b_{11}b_{22}b_{33}b_{44}$ Now the literal product $b_{11}b_{22}b_{33}b_{44}$ is found in Table I_b precisely where the literal product $a_{11}a_{22}a_{33}a_{44}$ is found in Table I_a namely, in the first row Therefore in Table I_b the signed product in the first row and last column is $+ b_{11}b_{22}b_{33}b_{44}$ (not $- b_{11}b_{22}b_{33}b_{44}$) Hence the signed product $- a_{11}a_{32}a_{23}a_{44}$ in A equals the negative of the signed product $+ b_{11}b_{22}b_{33}b_{44}$ in B Similarly it is proved that the signed product $+ a_{21}a_{12}a_{43}a_{34}$ in Table I_a is the negative of the signed product $- b_{21}b_{42}b_{13}b_{34}$ in Table I_b and that $- a_{41}a_{22}a_{33}a_{14}$ is the negative of $+ b_{41}b_{32}b_{23}b_{14}$

The preceding signed products of I_a were special terms in A It will be proved now that the general term in A is the negative

of a term in B. The general term in A is $(-1)^p a_{i_11}a_{i_22}a_{i_33}a_{i_44}$, in which $i_1i_2i_3i_4$ is an arrangement of the numbers 1, 2, 3, 4, showing p inversions. By (10) and rearrangement of factors it follows that

$$(11) \qquad (-1)^p a_{i_11}a_{i_22}a_{i_33}a_{i_44} = (-1)^p b_{i_11}b_{i_23}b_{i_32}b_{i_44},$$

$$(12) \qquad (-1)^p a_{i_11}a_{i_22}a_{i_33}a_{i_44} = (-1)^p b_{i_11}b_{i_32}b_{i_23}b_{i_44}.$$

Now the literal product $b_{i_11}b_{i_32}b_{i_23}b_{i_44}$ occurs in Table I_b precisely where $a_{i_11}a_{i_32}a_{i_23}a_{i_44}$ occurs in Table I_a, namely, in the row for the arrangement $i_1i_3i_2i_4$. By lemma 2 the number of inversions which $i_1i_3i_2i_4$ shows is $p - 1$ or $p + 1$, since $i_1i_2i_3i_4$ shows p inversions. Also $(-1)^{p-1} = (-1)^{p+1}$. Hence $(-1)^{p-1}b_{i_11}b_{i_32}b_{i_23}b_{i_44}$ is a term in B. Thus by (12) and Table I_b the literal product $b_{i_11}b_{i_32}b_{i_23}b_{i_44}$ establishes a correspondence between the term $(-1)^p a_{i_11}a_{i_22}a_{i_33}a_{i_44}$ in A and the term $(-1)^{p-1}b_{i_11}b_{i_32}b_{i_23}b_{i_44}$ in B. By (12) and the fact that $(-1)^p = -(-1)^{p-1}$, it is true that the general term in A is the negative of its corresponding term in B.

It is to be noted especially that terms in A which have different arrangements of first subscripts correspond to terms in B which have different arrangements of first subscripts. Thus a one-to-one correspondence has been established between the 4! terms whose sum is A and the 4! terms whose sum is B, such that corresponding terms are negatives of each other. Therefore by lemma 1 it is true that $A = -B$.

In general, if any two columns of B are interchanged and the result is called C, then a one-to-one correspondence can be established such that corresponding terms are negatives of each other. Hence it can be proved that $C = -B$. Hence theorem 3 has been proved if $n = 4$.

THEOREM 3. *If the symbol of a determinant A is obtained from the symbol of a determinant B by interchanging two columns of the symbol of B, then $A = -B$.*

PROBLEMS

1. Construct the table like Table II which contains the arrangement 3 4 5 1 2. Construct such a table for the arrangement 3 4 2 1 5.

2. Construct the table like Table II which contains the arrangement 4 2 5 1 3. Construct such a table for the arrangement 5 2 4 1 3.

3. Let $n = 5$, and let A be obtained by interchanging the third and fifth columns of B. Find the term in A which has the literal product $a_{31}a_{42}a_{53}a_{14}a_{25}$ as a factor. Find the term in B which corresponds to this term in A. Show

that these terms are negatives of each other. Treat the literal products $a_{21}a_{42}a_{53}a_{34}a_{15}$ and $a_{11}a_{42}a_{63}a_{34}a_{25}$ similarly.

4. Proceed as in problem 3 if the first and third columns are interchanged and the literal products are $a_{41}a_{22}a_{53}a_{14}a_{35}$, $a_{41}a_{12}a_{53}a_{24}a_{15}$, $a_{51}a_{22}a_{13}a_{44}a_{35}$.

5. Let $n = 7$ and find the number of inversions for the arrangements 3 7 2 6 1 5 4 and 3 7 4 6 1 5 2. Tabulate these arrangements with appropriate intervening arrangements such that each arrangement in the table is obtained from the preceding one by one interchange of adjacent numbers. Find the number of inversions for each arrangement in the table.

6. Proceed as in problem 5 with the arrangements 6 1 4 2 3 7 5 and 6 3 4 2 1 7 5.

7. *Proceed as in problem 3 if $n = 7$,* the second and fifth columns are interchanged, and the literal products are

$$a_{21}a_{12}a_{53}a_{34}a_{75}a_{66}a_{47} \qquad a_{71}a_{12}a_{23}a_{54}a_{45}a_{66}a_{37} \qquad a_{61}a_{42}a_{13}a_{74}a_{55}a_{26}a_{37}$$

8. Proceed as in problem 3 if $n = 7$, the first and fourth columns are interchanged, *and the literal products are*

$$a_{51}a_{12}a_{63}a_{74}a_{35}a_{26}a_{47} \qquad a_{51}a_{72}a_{63}a_{14}a_{35}a_{26}a_{47} \qquad a_{41}a_{72}a_{23}a_{14}a_{35}a_{66}a_{57}$$

All the ideas in the following proof of theorem 3 if n is arbitrary have been used in the preceding proof if $n = 4$. Let the symbol for A be (8) and let the symbol for B be obtained if each letter a in (8) is replaced by the letter b. By hypothesis s and t are fixed but arbitrary positive integers such that $1 \leqq s < t \leqq n$. In the preceding proof $n = 4$, $s = 2$, $t = 3$. Also by hypothesis A is obtained by interchanging the columns numbered s and t in B. That is

$$
\begin{aligned}
& a_{is} = b_{it} \quad (i = 1, \ldots, n) \\
(13) \qquad & a_{it} = b_{is} \quad (i = 1, \ldots, n) \\
& a_{ij} = b_{ij} \quad (j \neq s,\ j \neq t,\ i = 1, \ldots, n)
\end{aligned}
$$

Now the typical term in the sum A is $(-1)^p a_{i_1 1} \cdots a_{i_s s} \cdots a_{i_t t} \cdots a_{i_n n}$ in which $i_1 \cdots i_s \cdots i_t \cdots i_n$ is an arrangement of $1, \ldots, n$ showing p inversions. Also by (13) it is true that

$$
(14) \quad (-1)^p a_{i_1 1} \cdots a_{i_s s} \cdots a_{i_t t} \cdots a_{i_n n} = (-1)^p b_{i_1 1} \cdots b_{i_s t} \cdots b_{i_t s} \cdots b_{i_n n}
$$

Hence by rearranging factors in the product on the right-hand side of this equation it is true that

$$
(15) \quad (-1)^p a_{i_1 1} \cdots a_{i_s s} \cdots a_{i_t t} \cdots a_{i_n n} = (-1)^p b_{i_1 1} \cdots b_{i_t s} \cdots b_{i_s t} \cdots b_{i_n n}
$$

It is to be noted especially that in these products the factors which are indicated by dots have their second subscripts in natural order; that is, the only disturbed subscripts are among the exhibited subscripts. Hence in the second product in (15) the second subscripts are in natural order. The list of first subscripts in this product, namely, $i_1 \cdots i_t \cdots i_s \cdots i_n$, is an arrangement of $1, \cdots, n$ which is obtained from the arrangement $i_1 \cdots i_s \cdots i_t \cdots i_n$ by interchanging i_s and i_t. By hypothesis the latter arrangement shows p inversions. Hence by lemma 2 the number u of inversions shown by the former arrangement differs from p by an odd integer. Therefore $(-1)^{p-1} = (-1)^u$. Therefore $(-1)^{p-1}b_{i_11} \cdots b_{i_ts} \cdots b_{i_st} \cdots b_{i_nn}$ is a term in B. By (15) the literal product $b_{i_11} \cdots b_{i_ts} \cdots b_{i_st} \cdots b_{i_nn}$ establishes a correspondence between the term $(-1)^p a_{i_11} \cdots a_{i_ss} \cdots a_{i_tt} \cdots a_{i_nn}$ in A and the term $(-1)^{p-1}b_{i_11} \cdots b_{i_ts} \cdots b_{i_st} \cdots b_{i_nn}$ in B, and corresponding terms are negatives of each other. It is to be noted especially that terms in A which have different arrangements of first subscripts correspond to terms in B which have different arrangements of first subscripts.

Thus a one-to-one correspondence has been established between the $n!$ terms whose sum is A and the $n!$ terms whose sum is B such that each term in the sum A is the negative of its corresponding term in the sum B. Therefore by lemma 1 it is true that $A = -B$. This completes the proof of theorem 3 if n is arbitrary.

If the symbol of a determinant B of order n has two columns which are identical and if the symbol of a second determinant A is formed from the symbol of B by interchanging these two columns, then the symbol of A is exactly the symbol of B. Therefore $A = B$. By theorem 3 it is true that $A = -B$. Hence $B = -B$, and therefore $B = 0$. This completes the proof of theorem 4.

THEOREM 4. *If two columns of the symbol of a determinant are identical, then the determinant is zero.*

PROBLEMS

1. Use theorem 3 to verify the statement which follows (65) in chapter 5.
2. Using theorem 4, show that the points $(1, -1)$ and $(-2, 7)$ lie on the locus of the equation $\begin{vmatrix} x & 1 & -2 \\ y & -1 & 7 \\ 1 & 1 & 1 \end{vmatrix} = 0$. Hence this is an equation of the straight line through these points.

3. Let (a_1, b_1), (a_2, b_2), (a_3, b_3) be three distinct points. Using theorem 4 show that (a_2, b_2) and (a_3, b_3) lie on the locus of the equation

$$\begin{vmatrix} x & a_2 & a_3 \\ y & b_2 & b_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Hence this is an equation of the straight line through these points. Also the three points are collinear if and only if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$.

4. Let (a, b), (a_2, b_2), (a_3, b_3) be three non-collinear points. Using theorem 4 show that each of these points lies on the locus of the equation

$$\begin{vmatrix} x^2 + y^2 & a^2 + b^2 & a_2^2 + b_2^2 & a_3^2 + b_3^2 \\ x & a & a_2 & a_3 \\ y & b & b_2 & b_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

Hence this is an equation of the circle through these points. Find a necessary and sufficient condition that (a, b) lie on this circle.

The second fundamental property of determinants will now be illustrated and proved if $n = 4$. Let A and B be two determinants with symbols (9). By hypothesis

first column of symbol of A is first row of symbol of B
second column of symbol of A is second row of symbol of B
third column of symbol of A is third row of symbol of B
fourth column of symbol of A is fourth row of symbol of B

This hypothesis is also expressed by the statement that A is obtained by interchanging rows and columns in the symbol of B. In terms of the elements of the determinants this hypothesis is

(16) $a_{i1} = b_{1i}$, $a_{i2} = b_{2i}$, $a_{i3} = b_{3i}$, $a_{i4} = b_{4i}$ $(i = 1, 2, 3, 4)$.

The conclusion in this fundamental property is that $A = B$. This will be proved by lemmas 1 and 2 and equations (16).

It will be proved that there is a one-to-one correspondence between the signed products in the last column of Table I_a and the signed products in the last column of Table I_b such that corresponding signed products are equal. For example consider the signed product $+a_{11}a_{22}a_{33}a_{44}$ in I_a. By (16) it is true that this equals $+b_{11}b_{22}b_{33}b_{44}$. This latter term obviously is a term in I_b. Again consider $-a_{21}a_{32}a_{43}a_{14}$ in I_a. By (16) this equals

$-b_{12}b_{23}b_{34}b_{41}$. By rearranging the factors in this latter term it follows that $-a_{21}a_{32}a_{43}a_{14} = -b_{41}b_{12}b_{23}b_{34}$. This last term is a signed product in Table I_b because the literal product $b_{41}b_{12}b_{23}b_{34}$ occurs in Table I_b exactly where $a_{11}a_{12}a_{23}a_{34}$ occurs in Table I_a, namely, in the nineteenth row. Again, $-a_{41}a_{12}a_{23}a_{34} = -b_{14}b_{21}b_{32}b_{43} = -b_{21}b_{32}b_{43}b_{14}$, by (16) and rearranging factors b. The first signed product is in the nineteenth row of I_a, and the last is in the ninth row of I_b.

It will be proved now that the general term in A corresponds to a term in B and that corresponding terms are equal. The general term in A is $(-1)^p a_{i_11}a_{i_22}a_{i_33}a_{i_44}$, in which $i_1i_2i_3i_4$ is an arrangement of 1, 2, 3, 4, showing p inversions. By (16) it is true that

(17) $$(-1)^p a_{i_11}a_{i_22}a_{i_33}a_{i_44} = (-1)^p b_{1i_1}b_{2i_2}b_{3i_3}b_{4i_4}.$$

The factors in the product on the right-hand side of (17) can be written in any order, since multiplication is commutative. Let them be written so that the second subscripts appear in the natural order, as was done in each of the numerical illustrations just considered. Then, as in each of the numerical illustrations, the first subscripts form another arrangement of 1, 2, 3, 4. This new arrangement, which is formed by the first subscripts, will be designated by $j_1j_2j_3j_4$. Thus (17) becomes

(18) $$(-1)^p a_{i_11}a_{i_22}a_{i_33}a_{i_44} = (-1)^p b_{j_11}b_{j_22}b_{j_33}b_{j_44}.$$

For instance, in the last numerical example above $i_1 = 4$, $i_2 = 1$, $i_3 = 2$, $i_4 = 3$, and $j_1 = 2$, $j_2 = 3$, $j_3 = 4$, $j_4 = 1$. Now the literal product $b_{j_11}b_{j_22}b_{j_33}b_{j_44}$ occurs in I_b. Let v be the number of inversions which the arrangement $j_1j_2j_3j_4$ shows Then the signed product $(-1)^v b_{j_11}b_{j_22}b_{j_33}b_{j_44}$ is a term in B. In the last numerical example above, $v = 3$ because 2 3 4 1 shows 3 inversions; also $p = 3$ because 4 1 2 3 shows 3 inversions. It will be proved in general that $v - p$ is an even integer. It will then follow that $(-1)^p = (-1)^v$, and hence, by (18), that the term $(-1)^p a_{i_11}a_{i_22}a_{i_33}a_{i_44}$ in A equals the term $(-1)^v b_{j_11}b_{j_22}b_{j_33}b_{j_44}$ in B. The literal product $b_{j_11}b_{j_22}b_{j_33}b_{j_41}$, which emerges from this term in A as in (18), establishes the correspondence.

The method of proving that in general $v - p$ is an even integer will be illustrated on the arrangements arising from the signed product $-a_{31}a_{12}a_{43}a_{24}$ in A. By (16)

(19) $$(-1)^3 a_{31}a_{12}a_{43}a_{24} = (-1)^3 b_{13}b_{21}b_{34}b_{42}.$$

Since p is the number of inversions which the arrangement 3 1 4 2 of first subscripts on the left hand side of (19) shows therefore the second subscripts on the right-hand side of (19) show p inversions Now the following tabulation

$$
(20)\qquad \begin{array}{c} 3\ 1\ 4\ 2 \\ 1\ 3\ 4\ 2 \\ 1\ 2\ 4\ 3 \\ 1\ 2\ 3\ 4 \end{array}
$$

of these second subscripts is such that each arrangement is obtained from the preceding by one interchange of numbers Other tabulations in which this is true and the last arrangement is 1 2 3 4 are possible If s is the number of arrangements under 3 1 4 2 in such a tabulation it will be proved that $p - s$ is an even integer By lemma 2 each step in the tabulation changes the number of inversions by an odd integer Therefore the number p of inversions shown by 3 1 4 2 differs from the number zero of inversions shown by 1 2 3 4 by the sum of these s odd integers If they are designated by $2c_1 + 1$ $2c_2 + 1$ $2c_s + 1$ their sum is $2(c_1 + c_2 + \quad + c_s) + s$ Hence $p - 0 = 2(c_1 + c_2 + \quad + c_s) + s$ and $p - s$ is an even integer

Each of the products

$$
(21)\qquad \begin{array}{c} (-1)^3 b_{13}b_{21}b_{34}b_{42} \\ (-1)^3 b_{21}b_{13}b_{34}b_{42} \\ (-1)^3 b_{21}b_{42}b_{34}b_{13} \\ (-1)^3 b_{21}b_{42}b_{13}b_{34} \end{array}
$$

is equal to the term on the right-hand side of (19) and the second subscripts in these products form the tabulation (20) Also the first subscripts form the tabulation

$$
(22)\qquad \begin{array}{c} 1\ 2\ 3\ 4 \\ 2\ 1\ 3\ 4 \\ 2\ 4\ 3\ 1 \\ 2\ 4\ 1\ 3 \end{array}
$$

By definition v is the number of inversions which the last arrangement in (22) shows. Since there are s steps in (20), there are s steps in (21) and s steps in (22). By the same method which was used to prove that $p - s$ is an even integer it is here proved that $v - s$ is an even integer. Since each of $p - s$ and $v - s$ is an even integer, it follows that their difference $v - p$ is an even integer.

This method will be used now to prove that, if v is the number of inversions shown by $j_1j_2j_3j_4$ in (18) and if p is the number of inversions shown by $i_1i_2i_3i_4$ in (18), then $v - p$ is an even integer. Now the second subscripts on the right-hand side of (17) form the arrangement $i_1i_2i_3i_4$. Under this arrangement a tabulation analogous to (20) can be formed. This can be done, for example, by moving whichever of i_1, i_2, i_3, i_4 is 1 into the extreme left position, then by moving whichever of i_1, i_2, i_3, i_4 is 2 into the second position, and then by moving 3 into the third position The number 4 will then be in the fourth position. By definition s is the number of arrangements in this tabulation under $i_1i_2i_3i_4$. Now a set of products analogous to (21) is constructed. The product at the top is the right-hand side of (17), and the second subscripts form the arrangements in the preceding tabulation. This induces a tabulation analogous to (22) of the first subscripts of the letters b. There are s arrangements under 1 2 3 4 in this tabulation. The last arrangement is $j_1j_2j_3j_4$ on the right-hand side of (18). By lemma 2, $p - s$ is an even integer and $v - s$ is an even integer. Hence $v - p$ is an even integer, and $(-1)^p = (-1)^v$. Therefore the general term in A, which is the left-hand side of (18), equals the term $(-1)^v b_{j_11}b_{j_22}b_{j_33}b_{j_44}$ in B.

Terms in A which have different arrangements of first subscripts correspond to terms in B which have different arrangements of first subscripts. Therefore a one-to-one correspondence has been established, such that corresponding terms are equal. Therefore $A = B$, by lemma 1. Hence theorem 5 has been proved if $n = 4$.

THEOREM 5. *If the symbol of a determinant A is obtained from the symbol of a determinant B by interchanging rows and columns in the symbol of B, then $A = B$.*

The proof of theorem 5 if n is arbitrary involves no new ideas. By hypothesis

(23) $$a_{ij} = b_{ji} \quad (i = 1, \cdots, n; \quad j = 1, \cdots, n).$$

Now if (23) are applied to the typical term $(-1)^p a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}$ of A and if the factors b are reordered so that the second subscripts are in normal order there results a new arrangement $j_1 j_2 \cdots j_n$ of first subscripts such that

$$(24) \qquad (-1)^p a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} = (-1)^p b_{1 i_1} b_{2 i_2} \cdots b_{n i_n}$$
$$= (-1)^p b_{j_1 1} b_{j_2 2} \cdots b_{j_n n}$$

By an argument similar to that involving (20) (21) (22) it is proved that since $i_1 i_2 \cdots i_n$ shows p inversions the number v of inversions shown by $j_1 j_2 \cdots j_n$ differs from p by an even integer Therefore $(-1)^v = (-1)^p$ Hence the signed product $(-1)^v b_{j_1 1} b_{j_2 2} \cdots b_{j_n n}$ which is in Table I_b in the same row as the arrangement $j_1 j_2 \cdots j_n$ equals the last term in (24) Furthermore terms in A which have different arrangements of first subscripts equal terms in B which have different arrangements of first subscripts Thus a one-to-one correspondence has been established between the terms whose sum is A and the terms whose sum is B such that corresponding terms are equal Therefore $A = B$ by lemma 1 This completes the proof of theorem 5

An important corollary of theorem 5 and theorem 3 will be proved now Let the symbol of a determinant A be obtained from the symbol of a determinant B by interchanging two rows of the symbol of B Let these be the rows numbered s and t Now consider a determinant E whose symbol is obtained from the symbol of B as follows

C is obtained by interchanging rows and columns in B
D is obtained by interchanging columns numbered s and t in C
E is obtained by interchanging rows and columns in D

Therefore $C = B$ $D = -C$ $E = D$ On the other hand the symbol of E is precisely the symbol of A Hence $A = -B$ This completes the proof of theorem 6

THEOREM 6 *If the symbol of a determinant A is obtained from the symbol of a determinant B by interchanging two rows of the symbol of B then $A = -B$*

PROBLEMS

1 Prove that if two rows of the symbol of a determinant are identical then the determinant is zero

2. Let $n = 5$ and A be obtained by interchanging rows and columns of B. Find the terms in B which correspond to each of the following terms in A: $+a_{31}a_{42}a_{53}a_{14}a_{25}$; $+a_{21}a_{42}a_{53}a_{34}a_{15}$; $-a_{11}a_{42}a_{53}a_{34}a_{25}$.

3. Proceed as in problem 2 for the terms in A whose first subscripts form the following arrangements: 4 3 1 5 2; 4 3 2 1 5; 5 4 1 2 3.

4. Proceed as in problems 2 if $n = 7$ and the first subscripts form the following arrangements: 2 1 5 3 7 6 4; 7 1 3 5 4 6 2; 6 4 1 7 5 2 3.

5. Proceed as in problem 2 if $n = 7$ and the first subscripts form the following arrangements: 3 1 4 7 5 2 6; 5 7 6 1 3 2 4; 4 7 2 1 3 6 5.

6. Using problem 1, show that $(-1, 5)$ and $(2, 7)$ lie on the locus of the equation $\begin{vmatrix} x & y & 1 \\ -1 & 5 & 1 \\ 2 & 7 & 1 \end{vmatrix} = 0$. Hence this is an equation of the straight line through these points Prove this fact also by using problem 3 on page 142 and theorem 5

7. State and prove a problem which is suggested by problem 4 on page 142 and in which the variables are in the first row of the symbol.

4. Expansion of determinants of order n. Let the symbol of a determinant A of order n be

$$(25) \qquad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Then the minor A_{11} of a_{11} is, by definition, the determinant of order $n-1$ whose symbol is

$$(26) \qquad \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}.$$

The minor A_{21} of a_{21} is, by definition, the determinant of order $n-1$ whose symbol is

$$(27) \qquad \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}.$$

In general, the *minor of the element* a_{ij}, which appears in the ith row and jth column of the symbol (25), is the determinant of order $n-1$ whose symbol is obtained from the symbol (25) by deleting the ith row and the jth column of (25) The minor of a_{ij} is designated by A_{ij}

The proofs of the fundamental facts about expansion of a determinant of order n use a fact about the minor A_{11} which is so important in later work that it is stated and proved now as a lemma The lemma will be proved first if $n = 4$ Then A_{11} has the symbol

$$(28)\qquad \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

By (41) of chapter 5

$$(29)\quad A_{11} = +a_{22}a_{33}a_{44} - a_{22}a_{43}a_{34} + a_{32}a_{43}a_{24} - a_{32}a_{23}a_{44} + a_{42}a_{23}a_{34} - a_{42}a_{33}a_{24}$$

Hence $a_{11}A_{11}$ is the number

$$(30)\quad +a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{43}a_{34} + a_{11}a_{32}a_{43}a_{24} - a_{11}a_{32}a_{23}a_{44} + a_{11}a_{42}a_{23}a_{34} - a_{11}a_{42}a_{33}a_{24}$$

Now by Table I of section 1 the sum of all those signed products, each of which has a_{11} as a factor is precisely the number (30) Thus the following lemma has been proved if $n = 4$

LEMMA 3 *If A is the determinant whose symbol is (25), then $a_{11}A_{11}$ equals the sum of all the terms in A each of which has a_{11} as a factor*

No new ideas are involved in the proof of lemma 3 if n is arbitrary Let T designate the sum of all the terms in A each of which has a_{11} as a factor Lemma 1 will be applied to conclude that $a_{11}A_{11} = T$ Thus, first it will be proved that there are $(n-1)!$ terms in T Then it will be proved that $a_{11}A_{11}$ is a sum of $(n-1)!$ terms Then a one-to-one correspondence will be established between the terms in these sums, such that corresponding terms are equal

By the general definition of a determinant the terms which are in A and which have a_{11} as a factor are the signed products

$$(31)\quad (-1)^p a_{11}a_{i_2 2} \quad a_{i_n n} \text{ in which } 1\ i_2 \quad i_n \text{ is an arrangement of } 1, 2, \quad , n, \text{ showing } p \text{ inversions}$$

Hence $i_2 \cdots i_n$ is an arrangement of $2, \cdots, n$. There are exactly $(n-1)!$ different arrangements of $2, \cdots, n$. Therefore there are $(n-1)!$ terms in T.

Next it will be proved that $a_{11}A_{11}$ is the sum of $(n-1)!$ terms. There are $n-1$ rows in the symbol (26) of A_{11}. Hence, as in (29) if $n = 4$, each signed product in the sum which is A_{11} has exactly $n-1$ double-subscripted factors a. If $k_2k_3 \cdots k_n$ is an arrangement of $2, 3, \cdots, n$, showing w inversions, then

$$(32) \qquad (-1)^w a_{k_22}a_{k_33} \cdots a_{k_nn}$$

is a signed product in A_{11}. There are exactly $(n-1)!$ different arrangements of $2, 3, \cdots, n$. Therefore A_{11} is the sum of the $(n-1)!$ signed products (32). Therefore, as in (30) if $n = 4$, it is true that $a_{11}A_{11}$ is the sum of the $(n-1)!$ terms of the type

(33) $(-1)^w a_{11}a_{k_22}a_{k_33} \cdots a_{k_nn}$, in which $k_2k_3 \cdots k_n$ is an arrangement of $2, 3, \cdots, n$, showing w inversions.

It will be proved next that each term (33) in $a_{11}A_{11}$ equals a term in T. If $n = 4$, this was proved by inspection of Table I. In general, it is proved as follows. Since 1 is less than each of $2, 3, \cdots, n$, it is true that (33) becomes

(34) $(-1)^w a_{11}a_{k_22}a_{k_33} \cdots a_{k_nn}$, in which $1\, k_2k_3 \cdots k_n$ is an arrangement of $1, 2, 3, \cdots, n$, showing w inversions.

But by the general definition of a determinant the signed product (34) is in A. Since a_{11} is a factor in (34), therefore (34) is in T. Therefore the term (33) in $a_{11}A_{11}$ equals the term (34) in T. It is to be noted especially that terms in $a_{11}A_{11}$ having distinct arrangements of first subscripts are equal to terms in T having distinct arrangements of first subscripts. Therefore a one-to-one correspondence has been established. This completes the proof of lemma 3 if n is arbitrary.

A particular case of expansion of determinants of order n will be proved now. This case has been illustrated and proved if $n = 3$ in (54) of chapter 5 and if $n = 4$ in section 1.

THEOREM 7. *If A is the determinant whose symbol is (25), then* $A = a_{11}A_{11} - a_{21}A_{21} + a_{31}A_{31} - \cdots + (-1)^{n-1}a_{n1}A_{n1}$. *Therefore*

$$(35) \qquad A = \sum_{i=1}^{n}(-1)^{i-1}a_{i1}A_{i1}.$$

To prove theorem 7 let T_i designate the sum of all the terms in A each of which has a_{i1} as a factor. Now lemma 3 states that $T_1 = (-1)^{1-1}a_{11}A_{11}$. It will be proved next that

$$(36) \qquad T_i = (-1)^{i-1}a_{i1}A_{i1} \quad (i = 1, \ldots, n)$$

Let B_i designate the determinant whose symbol is

$$(37) \qquad \begin{vmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i-1\,1} & a_{i-1\,2} & \cdots & a_{i-1\,n} \\ a_{i+1\,1} & a_{i+1\,2} & \cdots & a_{i+1\,n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \qquad (i > 1)$$

Now the symbol of B_i can be obtained from the symbol of A by a succession of $i - 1$ interchanges of adjacent rows. Thus in the symbol of A the ith and $(i - 1)$st rows are interchanged. In this new symbol the elements $a_{i1}, a_{i2}, \ldots, a_{in}$ form the $(i - 1)$st row. This row is interchanged in turn with each preceding row. Thus (37) is obtained after $i - 1$ interchanges. Hence by theorem 6 it is true that $B_i = (-1)^{i-1}A$. It is to be noted that the minor of the element a_{i1} standing in the upper left-hand corner of the symbol of B_i is precisely the minor A_{i1} of the element a_{i1} standing in the first column and ith row of the symbol of A. Now let lemma 3 be applied to B_i. Thus A, a_{11}, A_{11} in lemma 3 are replaced by B_i, a_{i1}, A_{i1} respectively. Hence the sum of all the terms in B_i each of which has a_{i1} as a factor equals $a_{i1}A_{i1}$. It has already been proved that $B_i = (-1)^{i-1}A$. Hence the sum of all the terms in B_i each of which has a_{i1} as a factor equals $(-1)^{i-1}$ times the sum of all the terms in A each of which has a_{i1} as a factor. Hence by the definition of T_i it is true that $a_{i1}A_{i1} = (-1)^{i-1}T_i$. Thus the proof of equation (36) is completed.

The proof of theorem 7 is completed as follows. By the definition of a determinant of order n each term in A has either a_{11} as a factor or a_{21} as a factor, $\ldots$, or a_{n1} as a factor. Also no term in A has two of $a_{11}, a_{21}, \ldots, a_{n1}$ as factors. Therefore each term in A is a term in T_1 or a term in T_2, $\ldots$, or a term in T_n

Also each term in A is a term in only one of $T_1, T_2, \cdots, T_n$. Hence $A = T_1 + T_2 + \cdots + T_n$. If the equations (36) are substituted in this equation, the result is (35).

The result stated in theorem 7 is referred to as the expansion of A by minors of the elements of the first column, or as the expansion of A by its first column. It will be proved next that A can be expanded by minors of the elements of an arbitrary column.

THEOREM 8. *If A is a determinant whose symbol is (25), and if t is an arbitrary but fixed integer such that $1 \leqq t \leqq n$, then*

$$A = \sum_{i=1}^{n} (-1)^{i+t} a_{it} A_{it}. \tag{38}$$

If $t = 1$, this result follows from (35), since $(-1)^{i+1} = (-1)^{i-1}$. If $t > 1$, let C_t designate the determinant whose symbol is

$$\begin{vmatrix} a_{1t} & a_{11} & \cdots & a_{1,t-1} & a_{1,t+1} & \cdots & a_{1n} \\ a_{2t} & a_{21} & \cdots & a_{2,t-1} & a_{2,t+1} & \cdots & a_{2n} \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ a_{nt} & a_{n1} & \cdots & a_{n,t-1} & a_{n,t+1} & \cdots & a_{nn} \end{vmatrix} \quad (t > 1). \tag{39}$$

Now the symbol of C_t can be obtained from the symbol (25) of A by a succession of $t - 1$ interchanges of the tth column of the symbol of A with the preceding columns. Hence by theorem 3 $C_t = (-1)^{t-1}A$. It is to be noted that the minor of the element a_{1t} standing in the upper left-hand corner of the symbol of C_t is precisely the minor A_{1t} of the element a_{1t} standing in the first row and tth column of the symbol (25); the minor of a_{2t} in C_t is the minor A_{2t} of a_{2t} in (25); $\cdots$; the minor of a_{nt} in C_t is the minor A_{nt} of a_{nt} in (25). Now theorem 7 will be applied to C_t. Thus, if a_{i1}, A_{i1} in theorem 7 are replaced by a_{it}, A_{it} respectively, then $C_t = \sum_{i=1}^{n} (-1)^{i-1} a_{it} A_{it}$. If both sides of this equation are multiplied by $(-1)^{t-1}$, the result is the equation $(-1)^{t-1} C_t = (-1)^{t-1} \sum_{i=1}^{n} (-1)^{i-1} a_{it} A_{it}$. It has been proved earlier that $A = (-1)^{t-1} C_t$. Also it is true that $(-1)^{i-1+t-1} = (-1)^{i+t}$. Hence (38) has been proved.

The following theorem 9 gives the expansion of a determinant of order n by minors of the elements of a row It is a corollary of theorem 5 and theorem 8 This result is also referred to as the expansion by a row

THEOREM 9 *If A is a determinant whose symbol is (25) and if s is an arbitrary but fixed integer such that $1 \leqq s \leqq n$ then*

$$A = \sum_{j=1}^{n} (-1)^{s+j} a_{sj} A_{sj} \tag{40}$$

PROBLEMS

1 Evaluate each of the following determinants by expansion by its second column Check by expanding each by its third column

$$\begin{vmatrix} 7 & 3 & 2 \\ 1 & -5 & -1 \\ -4 & 2 & 1 \end{vmatrix} \quad \begin{vmatrix} 2 & 1 & 4 \\ 1 & -5 & -1 \\ -4 & 2 & 1 \end{vmatrix} \quad \begin{vmatrix} 2 & 1 & 4 \\ 7 & 3 & 2 \\ -4 & 2 & 1 \end{vmatrix} \quad \begin{vmatrix} 2 & 1 & 4 \\ 7 & 3 & 2 \\ 1 & -5 & -1 \end{vmatrix}$$

2 Proceed as in problem 1 for the following determinants

$$\begin{vmatrix} -1 & 4 & 2 \\ 2 & 1 & 5 \\ 3 & 5 & 9 \end{vmatrix} \quad \begin{vmatrix} 5 & 1 & 7 \\ 2 & 1 & 5 \\ 3 & 5 & 9 \end{vmatrix} \quad \begin{vmatrix} 5 & 1 & 7 \\ -1 & 4 & 2 \\ 3 & 5 & 9 \end{vmatrix} \quad \begin{vmatrix} 5 & 1 & 7 \\ -1 & 4 & 2 \\ 2 & 1 & 5 \end{vmatrix}$$

3 Evaluate each of the following determinants by expansion by its second row Check by expanding each by its third column

$$\begin{vmatrix} 1 & -1 & 4 \\ -5 & 2 & -1 \\ 2 & 5 & 1 \end{vmatrix} \quad \begin{vmatrix} 2 & -1 & 4 \\ 1 & 2 & -1 \\ -4 & 5 & 1 \end{vmatrix} \quad \begin{vmatrix} 2 & 1 & 4 \\ 1 & -5 & -1 \\ -4 & 2 & 1 \end{vmatrix} \quad \begin{vmatrix} 2 & 1 & -1 \\ 1 & -5 & 2 \\ -4 & 2 & 5 \end{vmatrix}$$

4 Proceed as in problem 3 for the following determinants

$$\begin{vmatrix} 2 & 1 & 7 \\ 3 & 4 & 2 \\ 1 & 5 & 9 \end{vmatrix} \quad \begin{vmatrix} 5 & 1 & 7 \\ -1 & 4 & 2 \\ 3 & 5 & 9 \end{vmatrix} \quad \begin{vmatrix} 5 & 2 & 7 \\ -1 & 3 & 2 \\ 3 & 1 & 9 \end{vmatrix} \quad \begin{vmatrix} 5 & 2 & 1 \\ -1 & 3 & 4 \\ 3 & 1 & 5 \end{vmatrix}$$

5 Let D be the determinant whose symbol is

$$\begin{vmatrix} 2 & 1 & -1 & 4 \\ 7 & 3 & 1 & 2 \\ 1 & -5 & 2 & -1 \\ -4 & 2 & 5 & 1 \end{vmatrix}$$

First evaluate D by expansion by its third column and use of the results of problem 1 Then evaluate D by expansion by its second row and use of the results of problem 3

6. Let D be the determinant whose symbol is

$$\begin{vmatrix} 5 & 2 & 1 & 7 \\ -1 & 3 & 4 & 2 \\ 2 & -4 & 1 & 5 \\ 3 & 1 & 5 & 9 \end{vmatrix}.$$

First evaluate D by expansion by its second column and use of the results of problem 2. Then evaluate D by expansion by its third row and use of the results of problem 4.

7. Evaluate the following determinant, first by expansion by its third row and then by expansion by its last column. Why is one of these methods preferable to the other? The symbol is

$$\begin{vmatrix} 1 & 2 & -1 & 3 \\ 2 & -5 & 3 & 7 \\ -1 & 4 & 0 & 1 \\ 9 & 1 & 2 & 1 \end{vmatrix}.$$

8. Proceed as in problem 7 for the determinant whose symbol is

$$\begin{vmatrix} 2 & 7 & -1 & 1 \\ 1 & 5 & -9 & 0 \\ 3 & 1 & 2 & 5 \\ -1 & 3 & 2 & 4 \end{vmatrix}.$$

5. Other properties of determinants of order n. One important property of determinants of order n will be proved now if $n = 4$. Let A and B be determinants of order 4 whose symbols are

$$(41) \qquad \begin{vmatrix} a_{11} & \cdots & a_{14} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{41} & \cdots & a_{44} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} b_{11} & \cdots & b_{14} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ b_{41} & \cdots & b_{44} \end{vmatrix}.$$

Let m be any number, and let $b_{11} = ma_{11}$, $b_{21} = ma_{21}$, $b_{31} = ma_{31}$, $b_{41} = ma_{41}$. Also let each other element of B equal the similarly situated element of A. In terms of the elements the hypothesis is that

$$(42) \qquad \begin{aligned} b_{i1} &= ma_{i1} \quad (i = 1, 2, 3, 4), \\ b_{ij} &= a_{ij} \quad (i = 1, 2, 3, 4;\ j = 2, 3, 4). \end{aligned}$$

It is to be noted that the minor B_{11} of b_{11} has precisely the same symbol that the minor A_{11} of a_{11} has. In general

$$(43) \qquad B_{i1} = A_{i1} \quad (i = 1, 2, 3, 4).$$

The expansion of A and B by their first columns is

$$(44)\qquad \begin{aligned} A &= a_{11}A_{11} - a_{21}A_{21} + a_{31}A_{31} - a_{41}A_{41} \\ B &= b_{11}B_{11} - b_{21}B_{21} + b_{31}B_{31} - b_{41}B_{41} \end{aligned}$$

By (42) and (43) equations (44) become

$$(45)\qquad \begin{aligned} A &= a_{11}A_{11} - a_{21}A_{21} + a_{31}A_{31} - a_{41}A_{41} \\ B &= ma_{11}A_{11} - ma_{21}A_{21} + ma_{31}A_{31} - ma_{41}A_{41} \end{aligned}$$

The right-hand side of the last equation in (45) is $m(a_{11}A_{11} - a_{21}A_{21} + a_{31}A_{31} - a_{41}A_{41})$. Hence $B = mA$. This fact which has been proved can be written in the form

$$(46)\qquad \begin{vmatrix} ma_{11} & a_{12} & a_{13} & a_{14} \\ ma_{21} & a_{22} & a_{23} & a_{24} \\ ma_{31} & a_{32} & a_{33} & a_{34} \\ ma_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = m \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

In general let A and B be determinants of order n whose symbols are respectively

$$(47)\qquad \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix}$$

Let m be any number and let t be an arbitrary but fixed integer such that $1 \leq t \leq n$. Let each element in the tth column of B be m times the corresponding element in the tth column of A and let each other element in B equal its corresponding element in A. This hypothesis will be expressed by the statement that a column of B is m times the corresponding column of A. The following theorem will be proved now if n is arbitrary.

THEOREM 10. *If a column of B is m times the corresponding column of A, then $B = mA$.*

No new ideas are involved in the proof of this theorem if n is arbitrary. By hypothesis

$$(48)\qquad \begin{aligned} b_{it} &= ma_{it} \quad (i = 1, 2, \cdots, n) \\ b_{ij} &= a_{ij} \quad (i = 1, 2, \cdots, n;\ j \neq t) \end{aligned}$$

Now, by theorem 8 applied to A and to B, it is true that

$$(49)\qquad \begin{aligned} A &= \sum_{i=1}^{n} (-1)^{i+t} a_{it} A_{it}, \\ B &= \sum_{i=1}^{n} (-1)^{i+t} b_{it} B_{it}. \end{aligned}$$

By equations (48_2) it is true that

$$(50)\qquad A_{it} = B_{it} \quad (i = 1, 2, \cdots, n).$$

If (48_1) and (50) are used in (49_2), it follows that

$$(51)\qquad B = \sum_{i=1}^{n} (-1)^{i+t} m a_{it} A_{it} = m \sum_{i=1}^{n} (-1)^{i+t} a_{it} A_{it}.$$

Hence by (49_1) it is true that $B = mA$.

A theorem analogous to theorem 10 will now be proved for rows. Let s be an arbitrary but fixed integer such that $1 \leqq s \leqq n$. Let $b_{sj} = ma_{sj}$ $(j = 1, \cdots, n)$, $b_{ij} = a_{ij}$ $(j = 1, \cdots, n;\ i \neq s)$. This hypothesis will be expressed by the statement that a row of B is m times the corresponding row of A. The auxiliary determinant C is the determinant whose symbol is obtained from the symbol of A by interchanging rows and columns. The auxiliary determinant D is similarly obtained from B. Then by theorem 5 it is true that $C = A$ and $D = B$. Now, by theorem 10 applied to C and D, it is true that $D = mC$. Hence $B = mA$. This completes the proof of theorem 11.

THEOREM 11. *If a row of B is m times the corresponding row of A, then $B = mA$.*

Determinants may be added, since determinants are merely numbers. However, in the following important case this addition may be accomplished merely by using the symbols of the determinants. This case will be illustrated now if $n = 4$. Let A and B be two determinants of order 4, with symbols (41). By hypothesis let

$$(52)\qquad b_{ij} = a_{ij} \quad (j = 2, 3, 4;\quad i = 1, 2, 3, 4).$$

It is to be noted especially that no relation is assumed between the elements of the first column of A and the elements of the first

No new ideas are involved in the proof of theorem 12 if n is arbitrary. By hypothesis

$$b_{ij} = a_{ij} \quad (j \neq t; \quad i = 1, \cdots, n). \tag{60}$$

Let C be an auxiliary determinant whose symbol is

$$\begin{vmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{n1} & \cdots & c_{nn} \end{vmatrix}.$$

Expansion of A, B, C by the tth column of each gives

$$\begin{aligned} A &= \sum_{i=1}^{n} (-1)^{i+t} a_{it} A_{it}, \\ B &= \sum_{i=1}^{n} (-1)^{i+t} b_{it} B_{it}, \\ C &= \sum_{i=1}^{n} (-1)^{i+t} c_{it} C_{it}. \end{aligned} \tag{61}$$

Now let

$$\begin{aligned} c_{it} &= a_{it} + b_{it} \quad (i = 1, \cdots, n), \\ c_{ij} &= a_{ij} \qquad\qquad (j \neq t; \quad i = 1, \cdots, n). \end{aligned} \tag{62}$$

By (60) and (62_2), it is true that

$$B_{it} = A_{it}, \quad C_{it} = A_{it} \quad (i = 1, \cdots, n). \tag{63}$$

Substitution from (63) and (62_1) in (61) yields

$$\begin{aligned} A &= \sum_{i=1}^{n} (-1)^{i+t} a_{it} A_{it}, \\ B &= \sum_{i=1}^{n} (-1)^{i+t} b_{it} A_{it}, \\ C &= \sum_{i=1}^{n} (-1)^{i+t} (a_{it} + b_{it}) A_{it}. \end{aligned} \tag{64}$$

Hence $C = A + B$. This completes the proof of theorem 12 for columns. The statement in theorem 12 about rows follows from that about columns by theorem 5.

column of B Since (52) are precisely (42_2), it follows that (43) and (44) are true, and hence

$$(53)\quad A + B = (a_{11} + b_{11})A_{11} - (a_{21} + b_{21})A_{21} + (a_{31} + b_{31})A_{31} - (a_{41} + b_{41})A_{41}$$

Let C be an auxiliary determinant whose symbol is

$$(54)\quad \begin{vmatrix} c_{11} & \cdot & c_{14} \\ & & \\ & & \\ c_{41} & & c_{44} \end{vmatrix}$$

Expansion of C by minors of the elements of its first column gives

$$(55)\quad C = c_{11}C_{11} - c_{21}C_{21} + c_{31}C_{31} - c_{41}C_{41}$$

Now let

$$(56)\quad \begin{aligned} c_{i1} &= a_{i1} + b_{i1} \quad (i = 1, 2, 3, 4), \\ c_{ij} &= a_{ij} \qquad (j \neq 1, \quad i = 1, 2, 3, 4) \end{aligned}$$

By (56_2) it is true that

$$(57)\quad C_{i1} = A_{i1} \quad (i = 1, 2, 3, 4)$$

Hence by (57) and (56_1) and (55) it is true that

$$(58)\quad C = (a_{11} + b_{11})A_{11} - (a_{21} + b_{21})A_{21} + (a_{31} + b_{31})A_{31} - (a_{41} + b_{41})A_{41}$$

By (58) and (53) it is true that

$$(59)\quad C = A + B$$

This fact which has been proved can be displayed in the form

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} & a_{14} \\ b_{21} & a_{22} & a_{23} & a_{24} \\ b_{31} & a_{32} & a_{33} & a_{34} \\ b_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} & a_{14} \\ a_{21}+b_{21} & a_{22} & a_{23} & a_{24} \\ a_{31}+b_{31} & a_{32} & a_{33} & a_{34} \\ a_{41}+b_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

It follows from the preceding proof and theorem 5 that an analogous statement is true of two determinants of order 4 whose symbols have corresponding elements equal in the second, third, and fourth rows This completes the proof of theorem 12 if $n = 4$, $t = 1$

No new ideas are involved in the proof of theorem 12 if n is arbitrary. By hypothesis

$$b_{ij} = a_{ij} \quad (j \neq t; \quad i = 1, \cdots, n). \tag{60}$$

Let C be an auxiliary determinant whose symbol is

$$\begin{vmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{n1} & \cdots & c_{nn} \end{vmatrix}.$$

Expansion of A, B, C by the tth column of each gives

$$(61) \qquad \begin{aligned} A &= \sum_{i=1}^{n} (-1)^{i+t} a_{it} A_{it}, \\ B &= \sum_{i=1}^{n} (-1)^{i+t} b_{it} B_{it}, \\ C &= \sum_{i=1}^{n} (-1)^{i+t} c_{it} C_{it}. \end{aligned}$$

Now let

$$(62) \qquad \begin{aligned} c_{it} &= a_{it} + b_{it} \quad (i = 1, \cdots, n), \\ c_{ij} &= a_{ij} \qquad\qquad (j \neq t; \quad i = 1, \cdots, n). \end{aligned}$$

By (60) and (62_2), it is true that

$$B_{it} = A_{it}, \quad C_{it} = A_{it} \quad (i = 1, \cdots, n). \tag{63}$$

Substitution from (63) and (62_1) in (61) yields

$$(64) \qquad \begin{aligned} A &= \sum_{i=1}^{n} (-1)^{i+t} a_{it} A_{it}, \\ B &= \sum_{i=1}^{n} (-1)^{i+t} b_{it} A_{it}, \\ C &= \sum_{i=1}^{n} (-1)^{i+t} (a_{it} + b_{it}) A_{it}. \end{aligned}$$

Hence $C = A + B$. This completes the proof of theorem 12 for columns. The statement in theorem 12 about rows follows from that about columns by theorem 5.

THEOREM 12 *Let A and B be determinants of order n Let t be an arbitrary but fixed integer such that $1 \leq t \leq n$ If each element in the symbol of B which is not in the tth column equals the corresponding element in the symbol of A then $A + B$ is indeed a determinant The symbol of the sum $A + B$ is obtained from the symbol of A by replacing each element in the tth column of A by the sum of this element and its corresponding element in the tth column of B The statement which is obtained from the preceding sentences by replacing the word column by the word row is also true*

PROBLEMS

In problems 1 2 3 4 7 8 prove the stated equalities

1 $\begin{vmatrix} 2 & 3 & -2 & -3 \\ -1 & 7 & 2 & 4 \\ 3 & -6 & 12 & 9 \\ 2 & 1 & -10 & 5 \end{vmatrix} - 3 \begin{vmatrix} 2 & 3 & -2 & -3 \\ -1 & 7 & 2 & 4 \\ 1 & -2 & 4 & 3 \\ 2 & 1 & -10 & 5 \end{vmatrix}$

2 $\begin{vmatrix} -1 & 2 & 10 & 7 \\ 2 & 1 & 20 & 1 \\ 3 & 1 & -5 & 2 \\ 1 & 7 & 0 & 3 \end{vmatrix} - 5 \begin{vmatrix} -1 & 2 & 2 & 7 \\ 2 & 1 & 4 & -1 \\ 3 & 1 & 1 & 2 \\ 1 & 7 & 0 & 3 \end{vmatrix}$

3 $\begin{vmatrix} 2 & 7\cdot 2 & -2 & -3 \\ -1 & 7(-1) & 2 & 4 \\ 1 & 7\cdot 1 & 4 & 3 \\ 2 & 7\cdot 2 & -10 & 5 \end{vmatrix} = 0$

4 $\begin{vmatrix} -1 & 2 & 2 & 7 \\ 4\cdot 3 & 4\cdot 1 & 4(-1) & 4\cdot 2 \\ 3 & 1 & -1 & 2 \\ 1 & 7 & 0 & 3 \end{vmatrix} = 0$

5 Write the following determinant as a sum of two determinants

$$\begin{vmatrix} 2 & 3 + 7\cdot 2 & -2 & -3 \\ -1 & 7 + 7(-1) & 2 & 4 \\ 1 & -2 + 7\cdot 1 & 4 & 3 \\ 2 & 1 + 7\cdot 2 & -10 & 5 \end{vmatrix}$$

6 Write $\begin{vmatrix} -1 & 2 & 2 & 7 \\ 2 + 4\cdot 3 & 1 + 4\cdot 1 & 4 + 4(-1) & -1 + 4\cdot 2 \\ 3 & 1 & -1 & 2 \\ 1 & 7 & 0 & 3 \end{vmatrix}$ as a sum

7 $\begin{vmatrix} 2 & 3 & -2 & -3 \\ -1 & 7 & 2 & 4 \\ 1 & -2 & 4 & 3 \\ 2 & 1 & -10 & 5 \end{vmatrix} - \begin{vmatrix} 2 & 3 + 7\cdot 2 & -2 & -3 \\ -1 & 7 + 7(-1) & 2 & 4 \\ 1 & -2 + 7\cdot 1 & 4 & 3 \\ 2 & 1 + 7\cdot 2 & -10 & 5 \end{vmatrix}$

8. $$\begin{vmatrix} -1 & 2 & 2 & 7 \\ 2 & 1 & 4 & -1 \\ 3 & 1 & -1 & 2 \\ 1 & 7 & 0 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 2 & 7 \\ 2+4\cdot 3 & 1+4\cdot 1 & 4+4(-1) & -1+4\cdot 2 \\ 3 & 1 & -1 & 2 \\ 1 & 7 & 0 & 3 \end{vmatrix}.$$

9. Apply theorems 10 and 11 to the determinants:

$$\begin{vmatrix} 1 & 0 & 10 & 5 \\ 3 & 1 & -4 & 7 \\ 4 & -2 & 2 & -1 \\ -3 & 3 & 6 & 9 \end{vmatrix}, \quad \begin{vmatrix} 2 & 1 & 5 & 2 \\ 6 & 7 & -1 & 1 \\ 10 & -8 & -2 & 4 \\ -2 & 0 & 1 & 3 \end{vmatrix}.$$

10. Apply theorems 10 and 11 to the determinants:

$$\begin{vmatrix} 2 & 7 & 2 & 1 \\ 15 & -5 & 10 & 0 \\ 7 & 3 & 8 & -1 \\ 3 & -1 & 6 & 4 \end{vmatrix}, \quad \begin{vmatrix} 1 & 3 & 2 & 3 \\ 1 & -1 & 4 & 2 \\ -2 & 2 & -6 & 10 \\ 5 & 0 & 8 & 7 \end{vmatrix}.$$

Another important property of determinants will be illustrated now if $n = 4$. Let A and B be determinants with symbols (41). Let m be any number. By hypothesis, let

$$(65) \qquad \begin{aligned} b_{ij} &= a_{ij} \quad (j = 2, 3, 4; \quad i = 1, 2, 3, 4), \\ b_{i1} &= a_{i1} + ma_{i3}. \end{aligned}$$

If $n = 4$ and $t = 1$ in theorem 12, then

$$B = A + \begin{vmatrix} ma_{13} & a_{12} & a_{13} & a_{14} \\ ma_{23} & a_{22} & a_{23} & a_{24} \\ ma_{33} & a_{32} & a_{33} & a_{34} \\ ma_{43} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

Hence by theorems 10 and 4

$$B = A + m\begin{vmatrix} a_{13} & a_{12} & a_{13} & a_{14} \\ a_{23} & a_{22} & a_{23} & a_{24} \\ a_{33} & a_{32} & a_{33} & a_{34} \\ a_{43} & a_{42} & a_{43} & a_{44} \end{vmatrix} = A + m\cdot 0 = A.$$

This completes the proof of the following theorem if $n = 4$, $s = 1$, $t = 3$.

THEOREM 13. *Let A and B be determinants of order n. Let m be any number. Let s and t be two distinct integers such that $1 \leqq s \leqq n$, $1 \leqq t \leqq n$. If each element in the sth column of B is the sum of the corresponding element in the sth column of A and the product of m and the corresponding element in the tth column of A,*

while each element not in the sth column of B equals the corresponding element in A, then B = A The statement obtained from this last statement by replacing the word column by the word row is also true

No new ideas are involved in the proof of theorem 13 if n is arbitrary By hypothesis

$$(66) \qquad \begin{aligned} b_{ij} &= a_{ij} \qquad (j \neq s \quad i = 1, \cdots, n), \\ b_{is} &= a_{is} + ma_{it} \quad (i = 1, \cdots, n) \end{aligned}$$

Let C be an auxiliary determinant with elements c_{ij} such that

$$(67) \qquad \begin{aligned} c_{ij} &= a_{ij} \quad (j \neq s \quad i = 1, \cdots, n), \\ c_{is} &= a_{it} \quad (i = 1, \cdots, n) \end{aligned}$$

Then by theorems 12 10 and 4 it is true that $B = A + mC$ Since $s \neq t$ the sth and tth columns of C are identical Hence $C = 0$ Hence $B = A$ The last statement of theorem 13 is a corollary of the preceding statement in theorem 13 and theorem 5

Theorems 10 11 12 13 are constantly used in the evaluation of determinants with numerical elements Thus if D is the determinant on the left-hand side of the equation in problem 1 in the set of problems on page 158 then a first step in the evaluation of D is indicated in that problem A second step is indicated in problem 7 Then in the determinant on the right-hand side of the equation in problem 7 the factor 2 would be removed and the second column simplified Remaining steps will be given now Thus

$$D = 6\begin{vmatrix} 2 & 17 & -1 & -3 \\ -1 & 0 & 1 & 4 \\ 1 & 5 & 2 & 3 \\ 2 & 15 & -5 & 5 \end{vmatrix} = 6\begin{vmatrix} 2 & 17 & -1+1\cdot 2 & -3 \\ -1 & 0 & 1+1(-1) & 4 \\ 1 & 5 & 2+1\cdot 1 & 3 \\ 2 & 15 & -5+1\cdot 2 & 5 \end{vmatrix}$$

$$= 6\begin{vmatrix} 2 & 17 & 1 & -3 \\ -1 & 0 & 0 & 4 \\ 1 & 5 & 3 & 3 \\ 2 & 15 & -3 & 5 \end{vmatrix} = 6\begin{vmatrix} 2 & 17 & 1 & -3+4\cdot 2 \\ -1 & 0 & 0 & 4+4(-1) \\ 1 & 5 & 3 & 3+4\cdot 1 \\ 2 & 15 & -3 & 5+4\cdot 2 \end{vmatrix}$$

$$= 6\begin{vmatrix} 2 & 17 & 1 & 5 \\ -1 & 0 & 0 & 0 \\ 1 & 5 & 3 & 7 \\ 2 & 15 & -3 & 13 \end{vmatrix}$$

Hence $D = 6(-1)^{2+1}(-1)A_{21}$, in which $A_{21} = \begin{vmatrix} 17 & 1 & 5 \\ 5 & 3 & 7 \\ 15 & -3 & 13 \end{vmatrix}$.

$$\text{Now } A_{21} = \begin{vmatrix} 17 & 1 & 5 \\ 5 + 1\cdot 15 & 3 + 1(-3) & 7 + 1\cdot 13 \\ 15 & -3 & 13 \end{vmatrix} = \begin{vmatrix} 17 & 1 & 5 \\ 20 & 0 & 20 \\ 15 & -3 & 13 \end{vmatrix}$$

$$= 20\begin{vmatrix} 17 & 1 & 5 \\ 1 & 0 & 1 \\ 15 & -3 & 13 \end{vmatrix} = 20\begin{vmatrix} 17 & 1 & 5 \\ 1 & 0 & 1 \\ 15 + 3\cdot 17 & -3 + 3\cdot 1 & 13 + 3\cdot 5 \end{vmatrix}$$

$$= 20\begin{vmatrix} 17 & 1 & 5 \\ 1 & 0 & 1 \\ 66 & 0 & 28 \end{vmatrix}. \text{ Hence } A_{21} = 20(-1)^{1+2}\cdot 1(28 - 66) = 760.$$

Therefore $D = 4560$. In practice many of the preceding steps are omitted, and often steps are taken simultaneously.

PROBLEMS

1. Complete the evaluation of the determinant on the left-hand side of problem 2 in the set of problems on page 158; a second step is indicated in problem 8.

2. Evaluate $\begin{vmatrix} 1 & 2 & -1 & 7 \\ 3 & 1 & 0 & 4 \\ 9 & 2 & 1 & 5 \\ -1 & 4 & 2 & 3 \end{vmatrix}$.

3. Evaluate D, D_1, D_2, D_3, D_4 for the equations

$$\begin{aligned} 3x + 5y - 7z + 13w &= 1, \\ -x + 2y + 5z + w &= 8, \\ 2x + y + z + 6w &= 2, \\ 5x + y + z + w &= 13. \end{aligned}$$

4. Evaluate D, D_1, D_2, D_3, D_4 for the equations

$$\begin{aligned} x + y + z - w &= 0, \\ 2x + y + 5z + 2w &= 3, \\ x - 5y - 4z + w &= -3, \\ 5x - 2y + z + 2w &= 4. \end{aligned}$$

5 Evaluate each of the determinants

$$\begin{vmatrix} 1 & 2 & -1 & 4 & 7 \\ 2 & 1 & 0 & 3 & 5 \\ 1 & 2 & 1 & 2 & 3 \\ -9 & 3 & 3 & 2 & -1 \\ 5 & -1 & -2 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & -5 & 4 & 3 \\ 4 & -3 & 9 & -1 & -4 \\ 7 & 2 & 5 & 11 & -1 \\ 2 & 0 & -2 & 1 & 4 \\ -1 & 7 & 3 & 2 & 5 \end{vmatrix}$$

6 Evaluate each of the determinants

$$\begin{vmatrix} -7 & 2 & -1 & 3 & -2 \\ -5 & 1 & 2 & 4 & 0 \\ -12 & 4 & 3 & 1 & 2 \\ 4 & -1 & -2 & 2 & 5 \\ -1 & 2 & 1 & 5 & -3 \end{vmatrix}, \quad \begin{vmatrix} 5 & 1 & 2 & -1 & 4 \\ -6 & 4 & -9 & -7 & 2 \\ -4 & 3 & 5 & -2 & 1 \\ -1 & 2 & 1 & 3 & 5 \\ 9 & -3 & 0 & 4 & -4 \end{vmatrix}$$

6. Laplace's development of a determinant of order n. Multiplication of determinants of order n In this section a rule will be explained by which any two determinants of the same order can be multiplied merely by operation on their symbols It will be found that the rule is more complicated than the rule for addition of determinants of the same order explained in theorem 12 It is to be noted, however that the hypothesis in the rule for multiplication of determinants is merely that the determinants be of the same order, whereas the hypothesis in theorem 12 is this condition and a condition of equality of certain of the corresponding elements in the two symbols Illustrations of the rule for multiplication of two determinants will be given In these illustrations the rule will be proved by actual multiplication Later the rule will be proved in general, not by actual multiplication, but by use of an important property of determinants This property is Laplace's development of a determinant It is analogous to the expansion property of theorem 8 and theorem 9

Let A and B be determinants whose symbols are

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix} \tag{68}$$

respectively Then $A = ad - cb$ and $B = a'd' - c'b'$ Therefore $AB = (ad - cb)(a'd' - c'b')$ Hence

$$AB = ada'd' - adc'b' - cba'd' + cbc'b' \tag{69}$$

Now let C be an auxiliary determinant whose symbol is

$$\begin{vmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{vmatrix} \tag{70}$$

Then $C = (aa' + bc')(cb' + dd') - (ca' + dc')(ab' + bd')$. Hence

(71) $$C = aa'dd' - dc'ab' - ca'bd' + bc'cb'.$$

By (69) and (71), $AB = C$. Therefore by (68) and (70)

(72) $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix} = \begin{vmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{vmatrix}.$$

This rule for multiplication of two determinants of order two is called the row-by-column rule of multiplication, because the elements in the rows of A are multiplied by the corresponding elements in the columns of B. Thus, corresponding to the elements a, b in the first row of A are respectively the elements a', c' in the first column of B. These corresponding elements give the products aa' and bc', whose sum $aa' + bc'$ is the element in the first row and first column of the product symbol. Again, corresponding to a, b in the first row of A are respectively b', d' in the second column of B. These corresponding elements give the products ab' and bd', whose sum is the element in the first row and second column of the product symbol. Similar statements can be made to explain the other elements in the product symbol.

The proof of the row-by-column rule of multiplication of two determinants of order three will be given now. Let A and B be

(73) $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}.$$

Then, as in (41) of chapter 5,

(74) $$A = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13},$$

(75) $$B = b_{11}b_{22}b_{33} - b_{11}b_{32}b_{23} + b_{21}b_{32}b_{13} - b_{21}b_{12}b_{33} + b_{31}b_{12}b_{23} - b_{31}b_{22}b_{13}.$$

Therefore AB can be found by multiplication of the expressions in (74) and (75). This result will contain 36 terms and need not be displayed here. Now let E be the auxiliary determinant formed from A and B by the row-by-column rule. Thus E has the symbol

(76) $$\begin{vmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{vmatrix}.$$

By the definition of a determinant of order three, E is a sum which can be obtained from (41) of chapter 5 by replacing a_{ij} there by $a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j}$ here The result can be simplified by performing the indicated operations It will be found that the final expression for E is precisely the same as the final expression for AB which was obtained by multiplying (74) and (75)

PROBLEMS

1 Find AB by (72) if $A = \begin{vmatrix} 2 & 5 \\ -3 & 7 \end{vmatrix}$, $B = \begin{vmatrix} -3 & 4 \\ 2 & 9 \end{vmatrix}$ Check by evaluating A and B and multiplying the results

2 Proceed as in problem 1 for $A = \begin{vmatrix} -8 & 7 \\ 3 & -2 \end{vmatrix}$, $B = \begin{vmatrix} 5 & 2 \\ 4 & 9 \end{vmatrix}$

3 Write and evaluate the symbol (76) if A and B are respectively

$$\begin{vmatrix} 2 & 3 & -7 \\ 9 & -2 & 5 \\ 3 & 5 & 4 \end{vmatrix}, \quad \begin{vmatrix} 5 & 7 & 2 \\ 5 & 9 & 8 \\ 4 & 3 & -2 \end{vmatrix}$$

Check by evaluating A and B and multiplying the results

4 Proceed as in problem 3 for

$$A = \begin{vmatrix} 4 & 2 & 3 \\ -7 & 5 & 2 \\ -3 & 9 & 8 \end{vmatrix} \quad B = \begin{vmatrix} 3 & 2 & -5 \\ 7 & -3 & 9 \\ 4 & 5 & 2 \end{vmatrix}$$

5 Proceed as in problem 3 for

$$A = \begin{vmatrix} 3 & 1 & 2 \\ 7 & -5 & 0 \\ 2 & -4 & 9 \end{vmatrix} \quad B = \begin{vmatrix} 3 & -2 & 2 \\ 5 & 8 & 1 \\ 0 & 7 & 4 \end{vmatrix}$$

6 Proceed as in problem 3 if

$$A = \begin{vmatrix} 1 & 5 & -3 \\ 7 & -2 & 2 \\ 0 & 1 & 9 \end{vmatrix}, \quad B = \begin{vmatrix} 2 & 0 & -5 \\ 3 & 7 & -1 \\ 4 & -3 & 8 \end{vmatrix}$$

7 Apply the row by-column rule to write the determinant symbol E of AB if

$$A = \begin{vmatrix} 2 & 1 & 0 & -1 \\ -2 & 3 & 5 & 2 \\ -1 & 2 & 1 & 4 \\ 4 & -1 & -1 & 3 \end{vmatrix} \quad \text{and} \quad B = \begin{vmatrix} 2 & 1 & -3 & 0 \\ 3 & -1 & 2 & 1 \\ -2 & 5 & -1 & 2 \\ 0 & -2 & 1 & 4 \end{vmatrix}$$

Check by evaluating A and B and multiplying the results

8 Proceed as in problem 7 if A and B are respectively

$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ -1 & 3 & 0 & 4 \\ 0 & 2 & 1 & -1 \\ 5 & 1 & 2 & 3 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & -1 & 5 & 2 \\ 0 & 2 & 7 & 1 \\ 2 & 3 & 1 & 0 \\ -1 & 1 & 2 & -1 \end{vmatrix}$$

The method of direct verification of the row-by-column rule will not be used in the proof of that rule for determinants of order n. In the general proof, however, Laplace's development of a determinant of order n and theorem 13 will be used.

A lemma which is basic in the proof of Laplace's development will now be illustrated if $n = 5$. Let D be the determinant whose symbol is

$$\begin{vmatrix} a_{11} & \cdots & a_{15} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{51} & \cdots & a_{55} \end{vmatrix}. \tag{77}$$

Therefore, by definition, D is a sum of 5! signed products. Among these 120 signed products the products

$$\begin{array}{ll} + a_{11}a_{22}a_{33}a_{44}a_{55} & - a_{21}a_{12}a_{33}a_{44}a_{55} \\ - a_{11}a_{22}a_{33}a_{54}a_{45} & + a_{21}a_{12}a_{33}a_{54}a_{45} \\ + a_{11}a_{22}a_{43}a_{54}a_{35} & - a_{21}a_{12}a_{43}a_{54}a_{35} \\ - a_{11}a_{22}a_{43}a_{34}a_{55} & + a_{21}a_{12}a_{43}a_{34}a_{55} \\ + a_{11}a_{22}a_{53}a_{34}a_{45} & - a_{21}a_{12}a_{53}a_{34}a_{45} \\ - a_{11}a_{22}a_{53}a_{44}a_{35} & + a_{21}a_{12}a_{53}a_{44}a_{35} \end{array} \tag{78}$$

occur. These twelve signed products have a very important common property, and this property is possessed by no other signed product of D.

This property is explained easily in terms of the new idea of complementary minors. Thus, the two minors

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix} \tag{79}$$

are complementary minors in D. Again,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_{23} & a_{24} & a_{25} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix} \tag{80}$$

are complementary minors in D. Again,

$$\begin{vmatrix} a_{31} & a_{32} \\ a_{51} & a_{52} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_{13} & a_{14} & a_{15} \\ a_{23} & a_{24} & a_{25} \\ a_{43} & a_{44} & a_{45} \end{vmatrix} \tag{81}$$

are complementary It is to be noted that ten two-rowed minors are formed from the first two columns of D They are the two-rowed minors in (79), (80), (81), and

$$
(82)\quad \begin{array}{llll}
\begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix}, & \begin{vmatrix} a_{11} & a_{12} \\ a_{51} & a_{52} \end{vmatrix}, & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, & \\
\begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix}, & \begin{vmatrix} a_{21} & a_{22} \\ a_{51} & a_{52} \end{vmatrix}, & \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}, & \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix}
\end{array}
$$

Now, if M is an arbitrary one of these ten two-rowed minors, then, by definition, *the minor C which is complementary to M in D* is the three-rowed minor which is obtained by deleting from D the two rows and the two columns in which M appear These ten pairs of complementary minors, determined by the first two columns of D, are very important

Now the common property of the twelve signed products (78) of D will be explained The minor $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ of D is a number, namely, the sum $a_{11}a_{22} - a_{21}a_{12}$ of the two terms $+ a_{11}a_{22}$ and $- a_{21}a_{12}$ Also the first term $+ a_{11}a_{22}$ is a factor in each signed product of the first column of (78), and the second term $- a_{21}a_{12}$ is a factor in each signed product of the second column of (78) On the other hand, if each signed product in D which is different from the twelve signed products (78) is displayed, it is found that no other signed product in D contains either $+ a_{11}a_{22}$ or $- a_{21}a_{12}$ as a factor Thus the common property that distinguishes the twelve signed products (78) in D is that each has as a factor one of the two terms whose sum is the minor $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

The lemma which is basic for the proof of Laplace's development of D concerns the signed products (78) and the particular complementary minors (79) Let M_1 designate the two-rowed minor in (79) and C_1 the three-rowed minor in (79) Thus C_1 is the complementary minor of M_1 in D Now, by definition,

$$(83)\qquad M_1 = a_{11}a_{22} - a_{21}a_{12}$$

$$(84)\qquad C_1 = a_{33}a_{44}a_{55} - a_{33}a_{54}a_{45} + a_{43}a_{54}a_{35} - a_{43}a_{34}a_{55} + a_{53}a_{34}a_{45} - a_{53}a_{44}a_{35}$$

By actual multiplication of (83) and (84) the product M_1C_1 is obtained The important fact is that this result is precisely the

sum of the twelve signed products (78). This completes the proof of the following basic lemma if $n = 5$, $k = 2$.

LEMMA 4. *Let n be an integer such that $n \geqq 4$. Let D be a determinant of order n. Let k be any integer such that $2 \leqq k \leqq n - 2$. Let M_1 be the k-rowed minor appearing in the upper left-hand corner of D. Let C_1 be the minor of D which is complementary to M_1. Then the sum of all the signed products in D, each of which has one of the terms of M_1 as a factor, equals M_1C_1.*

In the proof of lemma 4, if n is arbitrary, let U designate the sum of all the signed products of D, each of which has a term in M_1 as a factor. Since M_1 is a sum of terms and C_1 is a sum of terms, as in (83) and (84) if $n = 5$, therefore by actual multiplication M_1C_1 is a sum of terms. This last sum will be designated by V. A one-to-one correspondence will be established between the terms whose sum is U and the terms whose sum is V, such that corresponding terms are equal. By lemma 1 it will follow that $U = V$. This will complete the proof of lemma 4 if n is arbitrary.

First it will be proved that each term in V determines a unique term in U, and that different terms in V determine different terms in U. Let the symbol for D be $\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$. Then M_1 is the sum of the $k!$ signed products of the type

(85) $(-1)^p a_{i_11} \cdots a_{i_kk}$, in which $i_1 \cdots i_k$ is an arrangement of $1, \cdots, k$, showing p inversions.

Also, C_1 is the sum of the signed products of the type

(86) $(-1)^w a_{j_{k+1},k+1} \cdots a_{j_nn}$, in which $j_{k+1} \cdots j_n$ is an arrangement of $k + 1, \cdots, n$, showing w inversions.

Therefore each term in M_1C_1 is of the type

(87) $(-1)^p a_{i_11} \cdots a_{i_kk}(-1)^w a_{j_{k+1},k+1} \cdots a_{j_nn}$, in which the conditions stated in (85) and (86) hold.

Now (87) can be written in the form

(88) $$(-1)^{p+w}a_{i_1 1} \quad a_{i_k k}a_{j_{k+1}, k+1} \quad a_{j_n n}$$

Also, by the conditions stated in (85) and (86), $i_1 \quad i_k j_{k+1} \quad j_n$ is an arrangement of 1, , k, $k+1$, , n Finally, since each of i_1, , i_k is less than each of j_{k+1}, , j_n, the only inversions in $i_1 \quad i_k j_{k+1} \quad j_n$ are the inversions in $i_1 \quad i_k$ and those in $j_{k+1} \quad j_n$ Therefore

(89) $i_1 \quad i_k j_{k+1} \quad j_n$ is an arrangement

of 1, , n, showing $p + w$ inversions

Therefore (88) is a signed product in D Also (88) has the term (85) of M_1 as a factor Hence (88) is a term in U Thus it has been proved that each term in V determines a unique term in U Also, two distinct terms in V have distinct forms (87), and hence their corresponding terms (88) in U are distinct

The preceding argument also shows that the number n_V of terms in V is less than or equal to the number n_U of terms in U In order that lemma 1 may be used it will now be proved that $n_U = n_V$ It is sufficient to prove that there are no more terms in U than in V This will be done by showing that each term in U is determined by a term in V By the definition of U an arbitrary term in U is of the type

(90) $(-1)^q a_{i_1 1} \quad a_{i_k k}a_{i_{k+1} k+1} \quad a_{i_n n}$, in which $i_1 \quad i_n$ is an arrangement of 1, , n, showing q inversions, and

$i_1 \quad i_k$ is an arrangement of 1 k

Hence $i_{k+1} \quad i_n$ is an arrangement of $k+1$, , n If q_1 is the number of inversions appearing in $i_1 \quad i_k$ and if q_2 is the number of inversions appearing in $i_{k+1} \quad i_n$, then $q = q_1 + q_2$ Hence (90) becomes

(91) $$[(-1)^{q_1}a_{i_1 1} \quad a_{i_k k}][(-1)^{q_2}a_{i_{k+1}, k+1} \quad a_{i_n n}]$$

The first bracketed expression in (91) is a term in M_1, and the second bracketed expression in (91) is a term in C_1 Therefore the expression (91) is in V Also (91) determines (90) in the same way that (87) determined (88) Hence each term in U is determined by a term in V This completes the proof of lemma 4

PROBLEMS

1. Write the symbols for D, M_1, and C_1 if $n = 4$, $k = 2$ From Table I of section 1 write U, that is, the sum of all the signed products in D, each of which has a term of M_1 as a factor. Write M_1 as a sum of terms, and write C_1 as a sum of terms Multiply these results and thus find V. Check that $U = V$. This verifies lemma 4 if $n = 4$, $k = 2$

2. Verify lemma 4 if $n = 5$, $k = 3$

Another lemma which will be used in the proof of Laplace's development will now be illustrated if $n = 5$, $k = 2$. This will involve the determinant (77), the ten two-rowed minors from its first two columns, which were listed in (79), (80), (81), (82), and their complementary minors. Let M_2 designate the two-rowed minor in (80), and C_2 its complementary minor. All the signed products in D, each of which has as a factor one of the terms in M_2, could be displayed. Also expressions for M_2 and C_2, analogous to (83) and (84), could be displayed Then it could be verified that there are twelve signed products in D, each of which has as a factor one of the terms in M_2, and that their sum is $-M_2C_2$. However, this fact will be proved in an easier way. Let E designate the determinant

$$(92) \qquad \begin{vmatrix} a_{11} & \cdots & a_{15} \\ a_{31} & \cdots & a_{35} \\ a_{21} & \cdots & a_{25} \\ a_{41} & \cdots & a_{45} \\ a_{51} & \cdots & a_{55} \end{vmatrix}.$$

Therefore $E = -D$. It is to be noted that M_2 is the two-rowed minor in the upper left-hand corner of E and that the minor in E complementary to M_2 is precisely C_2. Hence, by lemma 4 with D, n, k, M_1, C_1 replaced respectively by E, 5, 2, M_2, C_2, it is true that the sum of all the signed products of E, each of which has one of the terms of M_2 as a factor, equals M_2C_2. Multiplication by -1 proves that the sum of the signed products of D, each of which has one of the terms of M_2 as a factor, equals $-M_2C_2$.

The analogous results for the remaining two-rowed minors of the first two columns of D could be proved separately. However, a general method, which will be used in the proof of lemma 5 if n is arbitrary, will now be illustrated. Let M be the minor $\begin{vmatrix} a_{21} & a_{22} \\ a_{51} & a_{52} \end{vmatrix}$ of D, and let C be its complementary minor. Let E

be the determinant

$$(93)\qquad \begin{vmatrix} a_{21} & a_{22} & a_{25} \\ a_{51} & a_{52} & a_{55} \\ a_{11} & a_{12} & a_{15} \\ a_{31} & a_{32} & a_{35} \\ a_{41} & a_{42} & a_{45} \end{vmatrix}$$

Thus E is obtained by the following sequence of interchanges

first and second rows of D to obtain D_1,

fifth and fourth rows of D_1 to obtain D_2,

fourth and third rows of D_2 to obtain D_3

third and second rows of D_3 to obtain E

It is especially to be noted that in forming the auxiliary determinant E from D interchanges of rows were made in such a way as to retain the relative positions of the rows of M and the rows of C. Since the first row of M was the row numbered 2 in D therefore *there were $2 - 1$ interchanges due to this row*. *Similarly there* were $5 - 2$ interchanges due to the second row of M. Hence $E = (-1)^{2-1+5-2}D$. Hence $E = D$. It follows as in the preceding proof that the sum of all the signed products of D each of which has a term of M as a factor equals $+MC$. It is to be noted that $(-1)^{2-1+5-2} = (-1)^{2+5+1+2}$ and that the integers 2 5 1 2 whose sum gives this last exponent are the numbers 2 and 5 of the rows of D in which the rows of M lie and the numbers 1 and 2 of the columns of D in which the columns of M lie.

This rule for obtaining the sign to be prefixed to the product MC gives the sign which was found previously in lemma 4 because the row numbers in M_1 are 1 and 2 and the column numbers are 1 and 2. Hence the new rule gives $(-1)^{1+2+1+2}M_1C_1$ this is the value in lemma 4. Again the result $-M_2C_2$ which was found earlier can be obtained by this rule. Thus the row numbers in M_2 are 1 and 3 and the column numbers are 1 and 2. Hence the new rule gives $(-1)^{1+3+1+2}M_2C_2$ this equals $-M_2C_2$ which was obtained earlier.

This general method will be illustrated again. Let M designate the minor $\begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix}$. Then form E by passing the fourth row of D over the $4 - 1$ preceding rows and in the result pass

the fifth row over all the preceding rows except the first, that is, over $5 - 2$ rows. Therefore $E = (-1)^{4-1+5-2}D$. Hence $E = (-1)^{4+5+1+2}D$. This method is applicable to the other two-rowed minors in (81) and (82). Therefore lemma 5 has been proved if $n = 5$, $k = 2$.

LEMMA 5. *Let n, k, D be as in lemma 4. Let M be a k-rowed minor in the first k columns of D. Let the rows of M lie in the rows of D which are numbered $i_1, \cdots, i_k$. Let C be the minor of D which is complementary to M. Then the sum of all the signed products of D, each of which has one of the terms of M as a factor, is* $(-1)^{i_1+\cdots+i_k+1+\cdots+k}MC$.

PROBLEMS

1. Verify lemma 5 if $n = 5$, $k = 2$, $M = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

2. Proceed as in problem 1 if $M = \begin{vmatrix} a_{31} & a_{32} \\ a_{51} & a_{52} \end{vmatrix}$.

3. Verify lemma 5 if $n = 4$, $k = 2$, $M = \begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix}$.

4. Proceed as in problem 3 if $M = \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$.

No new ideas are involved in the proof of lemma 5 if n and k are arbitrary. The auxiliary determinant E is formed from D by the following interchanges of rows: first, row i_1 of D is passed over the preceding $i_1 - 1$ rows; then row i_2 over all the preceding rows except the first, and hence over $i_2 - 2$ rows; $\cdots$; finally, row i_k over the preceding $i_k - k$ rows. Hence $E = (-1)^{i_1-1+i_2-2+\cdots+i_k-k}D$. Therefore $E = (-1)^{i_1+\cdots+i_k+1+\cdots+k}D$. Now, the k-rowed minor in the upper left-hand corner of E is M, and its complementary minor is C. Hence, by lemma 4 with D, M_1, C_1 replaced by E, M, C, and by multiplication by $(-1)^{i_1+\cdots+i_k+1+\cdots+k}$, the result stated in lemma 5 is obtained.

Laplace's development of the determinant (77) *by its first two columns* will now be proved. Let M_{is} designate the minor from the first two columns of D, whose rows appear in the ith and sth rows of D, and let C_{is} be the minor of D complementary to M_{is} Thus, M_{12} designates the two-rowed minor of (79), which was previously designated by M_1. Again, M_{13} designates the two-

rowed minor which was previously designated by M_2 and M_{25} designates the minor $\begin{vmatrix} a_{21} & a_{22} \\ a_{51} & a_{52} \end{vmatrix}$. It will be proved that

$$(94)\quad D = +M_{12}C_{12} - M_{13}C_{13} + M_{14}C_{14} - M_{15}C_{15} + M_{23}C_{23} - M_{24}C_{24} + M_{25}C_{25} + M_{34}C_{34} - M_{35}C_{35} + M_{45}C_{45}$$

Equation (94) can be written in the form

$$(95)\qquad D = \sum_{1 \leq i < s \leq 5} (-1)^{i+s+1+2} M_{is} C_{is}$$

This will be proved by establishing a one-to-one correspondence between the signed products whose sum is D and the terms which appear in the ten products on the right of (94). Now it was proved in the special proof of lemma 4 if $n = 5$ which preceded the proof if n is arbitrary that each term in $M_{12}C_{12}$ equals a unique signed product in D and in the special proof of lemma 5 if $n = 5$ that each term in $-M_{13}C_{13}$ *equals a unique signed product in* D. These facts illustrate the general fact, which follows from the general proof of lemma 5 that each term in each of the sums on the right-hand side of (94) equals a unique signed product in D. Also each term in M_{12} is distinct from each term in M_{13}. In general, if the numbers i, s are not the numbers j, t then each term in M_{is} is distinct from each term in M_{jt}. Hence all the terms in the ten products on the right-hand side of (94) are distinct.

It will now be proved that each signed product in D is determined by a term in one and only one of these ten sums. By definition of D an arbitrary signed product in (77) is of the form

(96) $(-1)^p a_{i_1 1} a_{i_2 2} a_{i_3 3} a_{i_4 4} a_{i_5 5}$ in which $i_1 \ldots i_5$ is an arrangement of 1 2 3 4, 5 showing p inversions

Now, either $a_{i_1 1} a_{i_2 2}$ or $-a_{i_1 1} a_{i_2 2}$ is a term in $M_{i_1 i_2}$. Hence the *signed product (96) is determined by a term in* $M_{i_1 i_2} C_{i_1 i_2}$ *on the* right-hand side of (94). A one-to-one correspondence such that corresponding terms are equal has been established between the *signed products whose sum is (77) and the terms in the ten products* on the right of (94). Therefore by lemma 1 their sums are equal. This completes the proof of (94). Thus the following *theorem has been proved if* $n = 5$, $k = 2$

THEOREM 14. *Let $i_1, \cdots, i_k$ be integers such that $k < n$, $1 \leq i_1 < \cdots < i_k \leq n$. Let D be a determinant of order n. Let $M_{[i]}$ designate the k-rowed minor of D whose columns appear in the first k columns of D and whose rows appear in the rows of D numbered $i_1, \cdots, i_k$. If $C_{[i]}$ designates its complementary minor, then*

$$(97) \qquad D = \sum_{1 \leq i_1 < \cdots < i_k \leq n} (-1)^{i_1 + \cdots + i_k + 1 + \cdots + k} M_{[i]} C_{[i]}.$$

No new ideas are involved in the proof of (97) if n and k are arbitrary. It has been proved in lemma 5 that each term in each of the products on the right of (97) equals a unique signed product in D. Also, each signed product in D is of the type

(98) $(-1)^p a_{i_1 1} \cdots a_{i_k k} a_{i_{k+1}, k+1} \cdots a_{i_n n}$, in which $i_1 \cdots i_k i_{k+1} \cdots i_n$ is an arrangement of $1, \cdots, n$, showing p inversions.

But $a_{i_1 1} \cdots a_{i_k k}$ or its negative is a term in $M_{[i]}$. Hence (98) is determined by a term in $M_{[i]} C_{[i]}$ on the right of (97). A one-to-one correspondence, such that corresponding terms are equal, has been established between the signed products in D and the terms on the right of (97). This completes the proof of theorem 14.

The Laplace development of the determinant (77) *by its third and fifth columns* will now be proved. Let M_{12} mean the minor $\begin{vmatrix} a_{13} & a_{15} \\ a_{23} & a_{25} \end{vmatrix}$; let M_{24} mean $\begin{vmatrix} a_{23} & a_{25} \\ a_{13} & a_{45} \end{vmatrix}$. In general, let M_{is} mean the minor of (77) whose columns are in the third and fifth columns of D and whose rows are in the rows numbered i and s. Let C_{is} designate the complementary minor of M_{is}. Form the auxiliary determinant F from D by first passing the third column over the preceding $3 - 1$ columns, and then passing the fifth column of this result over the preceding $5 - 2$ columns. Therefore $D = (-1)^{3-1+5-2} F$. Now the minor M_{is} of D appears in the first two columns of F, and the complementary minor to this minor in F is precisely the complementary minor C_{is} of M_{is} in D. Therefore, by (95) with D replaced by F, it is true that $F = \sum_{1 \leq i < s \leq 5} (-1)^{i+s+1+2} M_{is} C_{is}$. Multiplication by $(-1)^{3-1+5-2}$ and substitution from the preceding result gives

$$(99) \qquad D = \sum_{1 \leq i < s \leq 5} (-1)^{i+s+3+5} M_{is} C_{is}.$$

It is to be noted that the 3 and 5 in the exponent in (99) are the numbers of the columns for which the Laplace development is being obtained. In general, if j and t are fixed arbitrary integers such that $1 \leqq j < t \leqq 5$, then the Laplace development of (77) by its jth and tth columns is

$$(100) \qquad D = \sum_{1 \leqq r < s \leqq 5} (-1)^{r+s+j+t} M_{rs} C_{rs}.$$

Equation (100) will not be proved here because theorem 15 if $n = 5$, $k = 2$, $j_1 = j$, $j_2 = t$ gives (100), and theorem 15 if n is arbitrary will be proved next.

THEOREM 15. *Let $i_1, \ldots, i_k, j_1, \ldots, j_k$ be integers such that $k < n$, $1 \leqq i_1 < \cdots < i_k \leqq n$ and $1 \leqq j_1 < \cdots < j_k \leqq n$. Let D be a determinant of order n. Let $M_{[i][j]}$ designate the minor of D whose rows appear in the rows of D numbered $i_1, \ldots, i_k$ and whose columns appear in the columns of D numbered $j_1, \ldots, j_k$. Let $C_{[i][j]}$ designate its complementary minor. Then*

$$(101) \quad D = \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} (-1)^{i_1 + \cdots + i_k + j_1 + \cdots + j_k} M_{[i][j]} C_{[i][j]}.$$

In (101) $j_1, \ldots, j_k$ are fixed and $i_1, \ldots, i_k$ vary over all sets of integers such that $1 \leqq i_1 < \cdots < i_k \leqq n$.

PROBLEMS

1. Write the Laplace development if $n = 5$ and the columns are numbered 2, 4, 5; if the columns are numbered 1, 3, 4.
2. Proceed as in problem 1 if the columns are numbered 1, 3, 5; if the columns are numbered 2, 3, 4.
3. Write the Laplace development if $n = 6$ and the columns are numbered 1, 2, 5; if the columns are numbered 2, 4.
4. Proceed as in problem 3 if the columns are numbered 2, 3, 6; if the columns are numbered 3, 4.

No new ideas are involved in the proof of theorem 15 if n is arbitrary. Let the auxiliary determinant F be obtained by passing the column in D which is numbered j_1 over the preceding $j_1 - 1$ columns, then the column which is numbered j_2 over the preceding $j_2 - 2$ columns, . . . , and finally the column which is numbered j_k over the preceding $j_k - k$ columns. Therefore

$$(102) \qquad D = (-1)^{j_1 - 1 + j_2 - 2 + \cdots + j_k - k} F.$$

Now $M_{[i][j]}$ is in the first k columns of F and its complementary minor in F is precisely its complementary minor $C_{[i][j]}$ in D.

Hence by theorem 14

$$(103)\quad E = \sum_{1 \leq i_1 < \cdots < i_k \leq n} (-1)^{i_1+\cdots+i_k+1+\cdots+k} M_{[i],[j]} C_{[i],[j]}.$$

If (103) is multiplied by $(-1)^{j_1-1+\cdots+j_k-k}$ and (102) used, it is found that the result is precisely (101). This completes the proof of theorem 15 if n is arbitrary.

Theorem 15 is *Laplace's development of D by an arbitrary set of k of its columns. Laplace's development of D by an arbitrary set of k of its rows* is a corollary of theorem 5 and theorem 15. It is stated precisely as theorem 15 is stated except that (101) is replaced by

$$(104)\quad D = \sum_{1 \leq j_1 < \cdots < j_k \leq n} (-1)^{i_1+\cdots+i_k+j_1+\cdots+j_k} M_{[i],[j]} C_{[i],[j]},$$

and in the last sentence $i_1, \cdots, i_k$ are fixed and $j_1, \cdots, j_k$ vary.

PROBLEMS

1. Write the Laplace development if $n = 5$ and the rows are numbered 1, 3, 4; if the rows are numbered 2, 4, 5.
2. Proceed as in problem 1 if the rows are numbered 1, 2, 5; if the rows are numbered 3, 4, 5
3. Write the Laplace development if $n = 6$ and the rows are numbered 2, 3, 6; if the rows are numbered 1, 4, 5.
4. Proceed as in problem 3 if the rows are numbered 1, 3, 4; if the rows are numbered 2, 5, 6.

Laplace's development will now be used to prove the row-by-column rule of multiplication of determinants of order n. The proof will be given first if $n = 3$. It will be proved that the determinant E whose symbol is (76) is the product of the determinants A and B whose symbols are given in (73). Let G be the auxiliary determinant of order 6 whose symbol is

$$(105)\quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ -1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & -1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & -1 & b_{31} & b_{32} & b_{33} \end{vmatrix}.$$

First it will be proved that $G = AB$. Next it will be proved that the number G, for which (105) is the symbol, equals the number E, for which (76) is the symbol. It will follow that $E = AB$.

It is to be noted that the **3** and **5** in the exponent in (99) are the numbers of the columns for which the Laplace development is being obtained In general if j and t are fixed arbitrary integers such that $1 \leq j < t \leq 5$ then the Laplace development of (77) by its jth and tth columns is

$$D = \sum_{1 \leq r < s \leq 5} (-1)^{r+s+j+t} M_{rs} C_{rs} \tag{100}$$

Equation (100) will not be proved here because theorem 15 if $n = 5$ $k = 2$ $j_1 = j$ $j_2 = t$ gives (100) and theorem 15 if n is arbitrary will be proved next

THEOREM 15 *Let* i_1 i_k j_1 j_k *be integers such that* $k < n$ $1 \leq i_1 <$ $< i_k \leq n$ *and* $1 \leq j_1 <$ $< j_k \leq n$ *Let* D *be a determinant of order* n *Let* $M_{[i][j]}$ *designate the minor of* D *whose rows appear in the rows of* D *numbered* i_1 i_k *and whose columns appear in the columns of* D *numbered* j_1 j_k *Let* $C_{[i][j]}$ *designate its complementary minor Then*

$$D = \sum_{1 \leq i_1 < \quad < i_k \leq n} (-1)^{i_1 + \quad + i_k + j_1 + \quad + j_k} M_{[i][j]} C_{[i][j]} \tag{101}$$

In (101) j_1 j_k *are fixed and* i_1 i_k *vary over all sets of integers such that* $1 \leq i_1 <$ $< i_k \leq n$

PROBLEMS

1 Write the Laplace development if $n = 5$ and the columns are numbered 2 4 5 if the columns are numbered 1 3 4

2. Proceed as in problem 1 if the columns are numbered 1 3 5 if the columns are numbered 2 3 4

3 Write the Laplace development if $n = 6$ and the columns are numbered 1 2 5 if the columns are numbered 2 4

4 Proceed as in problem 3 if the columns are numbered 2 3 6 if the columns are numbered 3 4

No new ideas are involved in the proof of theorem 15 if n is arbitrary Let the auxiliary determinant F be obtained by passing the column in D which is numbered j_1 over the preceding $j_1 - 1$ columns then the column which is numbered j_2 over the preceding $j_2 - 2$ columns and finally the column which is numbered j_k over the preceding $j_k - k$ columns Therefore

$$D = (-1)^{j_1 - 1 + j_2 - 2 + \quad + j_k - k} F \tag{102}$$

Now $M_{[i][j]}$ is in the first k columns of F and its complementary minor in F is precisely its complementary minor $C_{[i][j]}$ in D

It is to be noted that G_3 was obtained from G by three steps. In each step an appropriate multiple of the first, second, or third column was added to the fourth column. Also the symbol (108) of G_3 and the symbol (105) of G are precisely the same except in the fourth column. Similarly from G_3 three more determinants can be obtained by adding appropriate multiples of the first, second, and third columns to the fifth column. The symbol of the sixth determinant G_6 is the same as (108) except that the fifth column is

$$\begin{matrix} a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ 0 \\ 0 \\ 0 \end{matrix} \tag{109}$$

Finally, three more determinants are obtained from G_6 by adding appropriate multiples of the first, second, and third columns to the sixth column. The symbol of the ninth determinant G_9 is the same as the symbol of G_6 except that the sixth column is

$$\begin{matrix} a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \\ 0 \\ 0 \\ 0 \end{matrix} \tag{110}$$

Therefore $G = G_9$.

Next, the Laplace development of G_9 by its last three columns will be written. In the last three columns each minor of order three, except the minor in the upper right-hand corner, has at least one row of zeros and hence is zero. This minor in the upper right-hand corner is E, and its complementary minor has the symbol $\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$. Therefore the Laplace development of G_9 is

$$G_9 = (-1)^{4+5+6+1+2+3}E\cdot(-1).$$

Therefore $G = E$. This completes the proof.

Thus will be completed this alternative proof of the row by column rule for multiplication of determinants of order three

The Laplace development of G by its first three rows is obtained from (104) if D is replaced by G and if $n = 6$ $k = 3$ $i_1 = 1$ $i_2 = 2$ $i_3 = 3$ Also by (105) it is found that in the first three rows of G each minor of order three except the minor in the upper left-hand corner, has at least one column of zeros and hence is zero Also the minor of order three in the upper left-hand corner is A, and its complementary minor is B Therefore $G = (-1)^{1+2+3+1+2+3}AB = AB$

Next let G_1 be the determinant whose symbol is obtained from the symbol (105) by adding to each element of the fourth column the product of b_{11} and the corresponding element of the first column By theorem 13 $G = G_1$ Also the symbol of G_1 is

$$(106) \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11}b_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{21}b_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{31}b_{11} & 0 & 0 \\ -1 & 0 & 0 & 0 & b_{12} & b_{13} \\ 0 & -1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & -1 & b_{21} & b_{32} & b_{33} \end{vmatrix}$$

Let G_2 be the determinant whose symbol is obtained from the symbol (106) of G_1 by adding to each element of the fourth column the product of b_{21} and the corresponding element of the second column Therefore $G_1 = G_2$ and the symbol of G_2 is

$$(107) \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11}b_{11} + a_{12}b_{21} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{21}b_{11} + a_{22}b_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{31}b_{11} + a_{32}b_{21} & 0 & 0 \\ -1 & 0 & 0 & 0 & b_{12} & b_{13} \\ 0 & -1 & 0 & 0 & b_{22} & b_{23} \\ 0 & 0 & -1 & b_{31} & b_{32} & b_{33} \end{vmatrix}$$

Let G_3 be obtained from G_2 by adding to the fourth column b_{31} times the third column Therefore $G_2 = G_3$ and G_3 is

$$(108) \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & 0 & 0 \\ -1 & 0 & 0 & 0 & b_{12} & b_{13} \\ 0 & -1 & 0 & 0 & b_{22} & b_{23} \\ 0 & 0 & -1 & 0 & b_{32} & b_{33} \end{vmatrix}$$

It is to be noted that G_3 was obtained from G by three steps. In each step an appropriate multiple of the first, second, or third column was added to the fourth column. Also the symbol (108) of G_3 and the symbol (105) of G are precisely the same except in the fourth column. Similarly from G_3 three more determinants can be obtained by adding appropriate multiples of the first, second, and third columns to the fifth column. The symbol of the sixth determinant G_6 is the same as (108) except that the fifth column is

$$
\begin{array}{c}
a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\
a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\
a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\
0 \\
0 \\
0
\end{array}
\tag{109}
$$

Finally, three more determinants are obtained from G_6 by adding appropriate multiples of the first, second, and third columns to the sixth column. The symbol of the ninth determinant G_9 is the same as the symbol of G_6 except that the sixth column is

$$
\begin{array}{c}
a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\
a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\
a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \\
0 \\
0 \\
0
\end{array}
\tag{110}
$$

Therefore $G = G_9$.

Next, the Laplace development of G_9 by its last three columns will be written. In the last three columns each minor of order three, except the minor in the upper right-hand corner, has at least one row of zeros and hence is zero. This minor in the upper right-hand corner is E, and its complementary minor has the symbol $\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$. Therefore the Laplace development of G_9 is

$$G_9 = (-1)^{4+5+6+1+2+3} E \cdot (-1).$$

Therefore $G = E$. This completes the proof.

No new ideas are involved in the proof of the row-by-column rule for multiplication of determinants of the same arbitrary order n. Let the symbols of A and B be

$$(111) \qquad \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix}$$

Let E be the determinant of order n having in its ith row and jth column the number

$$(112) \qquad a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Then the row-by-column rule states that $AB = E$.

The proof of this rule uses an auxiliary determinant G of order $2n$ whose symbol is

$$(113) \qquad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & b_{11} & \cdots & b_{1n} \\ 0 & -1 & \cdots & 0 & b_{21} & \cdots & b_{2n} \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & -1 & b_{n1} & \cdots & b_{nn} \end{vmatrix}$$

It is to be noted especially that in the lower left-hand corner of (113) there is an n-rowed minor in which each element in the principal diagonal is -1 and each other element is zero. In the upper right-hand corner there is an n-rowed minor consisting of zeros. Therefore in the Laplace development of G by its first n rows each term is zero except the term containing the minor in the upper left-hand corner. This minor is A, and its complementary minor B. Therefore $G = (-1)^{1+\cdots+n+1+\cdots+n}AB = AB$.

Now there is another auxiliary determinant H which is equal to G and which has a very simple Laplace development by its last n columns. This determinant H is the last in a sequence of equal

determinants obtained by taking $j = n + 1, \cdots, 2n$ in succession in the following operation:

multiply the first column by $b_{1,j-n}$ and add to the jth;
multiply the second column by $b_{2,j-n}$ and add to the jth;

. . . .
. . . .
. . . .

multiply the nth column by $b_{n,j-n}$ and add to the jth.

Then the first n columns of the symbol of H are precisely the first n columns of (113). In the last n rows of the last n columns of H each element is zero. The elements $h_{i,n+j}$ in the first n rows of the last n columns of H have $1 \leqq i \leqq n$ and $1 \leqq j \leqq n$. The element $h_{i,n+j}$ is precisely the number (112). Therefore the first n rows of the last n columns of H form the symbol of E. The minor of E in H is $(-1)^n$, since the last n rows of the first n columns of H are precisely as they are in (113). Since each element in the last n rows of the last n columns of H is zero, each n-rowed minor from the last n columns of H, except that in the first n rows, is zero. Therefore the Laplace development of H by its last n columns gives $H = (-1)^{1+2+\cdots+n+(n+1)+\cdots+2n}E(-1)^n$. By the rule for the sum of an arithmetic progression $1 + \cdots + 2n = n(2n + 1)$. Therefore the exponent of -1 is $n(2n + 1) + n$. Since this exponent is an even integer, it follows that $H = E$. It has been proved earlier that $G = H$ and $G = AB$. Therefore $AB = E$. This completes the proof of the row-by-column rule for multiplication of two determinants of arbitrary order n.

PROBLEMS

1. Using the row-by-column rule, multiply the two determinants in problem 5 in the list of problems on p. 164.

2. Proceed as in problem 1 for the two determinants in problem 6 in that list.

CHAPTER 7

SYSTEMS OF LINEAR EQUATIONS AND DETERMINANTS

1. Systems of n linear equations in n unknowns. General results, which are analogous to the results obtained in chapter 5 for three linear equations in three unknowns, will be obtained in this section. The methods of proof are simpler than those in chapter 5 because the general theorems of chapter 6 are available here. Let the n linear equations in n unknowns be

$$
(1)\qquad
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= k_1,\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= k_2,\\
&\;\;\vdots\\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= k_n
\end{aligned}
$$

Let D designate the determinant whose symbol is

$$
(2)\qquad
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n}\\
a_{21} & a_{22} & \cdots & a_{2n}\\
\vdots & \vdots & & \vdots\\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
$$

Let D_i designate the determinant whose symbol is obtained from (2) by replacing the ith column of (2) by the column of constants $k_1, k_2, \cdots, k_n$ in (1).

It will be proved that if $x_1, x_2, \cdots, x_n$ is an ordered set of numbers which satisfy (1), then

$$(3)\qquad Dx_1 = D_1,\ Dx_2 = D_2,\ \cdots,\ Dx_n = D_n$$

By definition the symbol of the determinant D_1 is

$$
(4)\qquad \begin{vmatrix} k_1 & a_{12} & \cdots & a_{1n} \\ k_2 & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ k_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}.
$$

Also, by hypothesis, equations (1) are true. Hence, by substitution from (1), the symbol (4) becomes

$$
(5)\qquad \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & a_{12} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & a_{22} & \cdots & a_{2n} \\ \cdot \quad\quad \cdot \quad\quad\quad \cdot & \cdot & & \cdot \\ \cdot \quad\quad \cdot \quad\quad\quad \cdot & \cdot & & \cdot \\ \cdot \quad\quad \cdot \quad\quad\quad \cdot & \cdot & & \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}.
$$

By theorem 12 of chapter 6 the determinant (5) equals the sum

$$
(6)\qquad \begin{vmatrix} a_{11}x_1 & a_{12} \cdots a_{1n} \\ a_{21}x_1 & a_{22} \cdots a_{2n} \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ a_{n1}x_1 & a_{n2} \cdots a_{nn} \end{vmatrix} + \begin{vmatrix} a_{12}x_2 + \cdots + a_{1n}x_n & a_{12} \cdots a_{1n} \\ a_{22}x_2 + \cdots + a_{2n}x_n & a_{22} \cdots a_{2n} \\ \cdot \quad\quad\quad \cdot & \cdot \quad\quad \cdot \\ \cdot \quad\quad\quad \cdot & \cdot \quad\quad \cdot \\ \cdot \quad\quad\quad \cdot & \cdot \quad\quad \cdot \\ a_{n2}x_2 + \cdots + a_{nn}x_n & a_{n2} \cdots a_{nn} \end{vmatrix}.
$$

The second determinant in (6) equals the following sum:

$$
(7)\qquad \begin{vmatrix} a_{12}x_2 & a_{12} \cdots a_{1n} \\ a_{22}x_2 & a_{22} \cdots a_{2n} \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ a_{n2}x_2 & a_{n2} \cdots a_{nn} \end{vmatrix} + \begin{vmatrix} a_{13}x_3 + \cdots + a_{1n}x_n & a_{12} \cdots a_{1n} \\ a_{23}x_3 + \cdots + a_{2n}x_n & a_{22} \cdots a_{2n} \\ \cdot \quad\quad\quad \cdot & \cdot \quad\quad \cdot \\ \cdot \quad\quad\quad \cdot & \cdot \quad\quad \cdot \\ \cdot \quad\quad\quad \cdot & \cdot \quad\quad \cdot \\ a_{n3}x_3 + \cdots + a_{nn}x_n & a_{n2} \cdots a_{nn} \end{vmatrix}
$$

Repetition of this process shows that D_1 is the sum of n determinants. Thus, let $B_1, B_2, \cdots, B_n$ be defined by

$$
(8)\qquad B_j = \begin{vmatrix} a_{1j}x_j & a_{12} & \cdots & a_{1n} \\ a_{2j}x_j & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{nj}x_j & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \qquad (j = 1, 2, \cdots, n).
$$

Then

$$(9) \qquad D_1 = B_1 + B_2 + \cdots + B_n$$

Now if C_j is defined by

$$(10) \qquad C_j = \begin{vmatrix} a_{1j} & a_{12} & \cdots & a_{1n} \\ a_{2j} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{nj} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad (j = 1, 2, \cdots, n),$$

then $B_j = x_j C_j$ Also $C_1 = D$ and $C_j = 0$ if $j > 1$ Hence

$$(11) \qquad B_1 = x_1 D \quad B_j = 0 \quad (j \neq 1)$$

Substitution of (11) in (9) proves (3_1) Similarly (3_2) is proved by using (1) in the symbol for D_2 Each of equations (3) is proved in this way

If $x_1 \cdots x_n$ is a solution of (1) and if $D \neq 0$ then from (3)

$$(12) \qquad x_1 = \frac{D_1}{D} \quad x_2 = \frac{D_2}{D} \quad \cdots, \quad x_n = \frac{D_n}{D}$$

This completes the proof of the following theorem

THEOREM 1 *Let D be the determinant of the coefficients of the n variables in the n linear equations (1) and let D_i be the determinant whose symbol is obtained from the symbol of D by replacing the ith column of the symbol of D by $k_1 \cdots k_n$ If $D \neq 0$ and if there is a solution of (1) then that solution is the ordered set of numbers D_1/D D_2/D $\cdots, D_n/D$*

It will be proved next that if $D \neq 0$ then the set of numbers D_1/D D_2/D $\cdots$ D_n/D is a solution of equations (1) These numbers satisfy the first equation in (1) if and only if $a_{11}(D_1/D) + a_{12}(D_2/D) + \cdots + a_{1n}(D_n/D) = k_1$ and hence if and only if $a_{11}D_1 + a_{12}D_2 + \cdots + a_{1n}D_n = k_1 D$ and hence if and only if

$$(13) \qquad k_1 D - a_{11}D_1 - a_{12}D_2 - \cdots - a_{1n}D_n = 0$$

Let E designate the number on the left-hand side of (13) It is to be proved that E is indeed the number zero By definition

$$(14) \qquad E = k_1 D - a_{11}D_1 - a_{12}D_2 - \cdots - a_{1n}D_n$$

Also, the number D_1 in (14) is the determinant of order n whose symbol is

$$\tag{15} \begin{vmatrix} k_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ k_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ k_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}.$$

Hence, by the definition of a determinant, D_1 is the sum of $n!$ signed products. Again, D_2 is the determinant of order n whose symbol is

$$\tag{16} \begin{vmatrix} a_{11} & k_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & k_2 & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_{n1} & k_n & a_{n3} & \cdots & a_{nn} \end{vmatrix},$$

and D_2 i a sum of $n!$ signed products. In general, each of D, $D_1, \cdots, D_n$ in (14) is a sum of $n!$ signed products. One way to evaluate the number (14) would be to substitute these sums in (14) and simplify the result. This method of proving that E is indeed zero would be very complicated. Another method would be to expand each of $D, D_1, \cdots, D_n$ by minors of a row or column and to substitute these expansions in (14). This method of proving that E is indeed zero would also be complicated.

A very simple method of proving that E is zero will be explained next. This method may seem less direct than the expansion methods because expansion of determinants has been used frequently. Let E_2 be defined by

$$\tag{17} E_2 = \begin{vmatrix} k_1 & a_{11} & a_{13} & \cdots & a_{1n} \\ k_2 & a_{21} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ k_n & a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix}.$$

Then

$$\tag{18} D_2 = -E_2.$$

Again by definition D_3 is the determinant whose symbol is

$$(19)\qquad \begin{vmatrix} a_{11} & a_{12} & k_1 & a_{14} & & a_{1n} \\ a_{21} & a_{22} & k_2 & a_{24} & & a_{2n} \\ & & & & & \\ & & & & & \\ a_{n1} & a_{n2} & k_n & a_{n4} & & a_{nn} \end{vmatrix}$$

Hence if E_3 is defined by

$$(20)\qquad E_3 = \begin{vmatrix} k_1 & a_{11} & a_{12} & a_{14} & & a_{1n} \\ k_2 & a_{21} & a_{22} & a_{24} & & a_{2n} \\ & & & & & \\ & & & & & \\ k_n & a_{n1} & a_{n2} & a_{n4} & & a_{nn} \end{vmatrix}$$

then

$$(21)\qquad D_3 = +E_3$$

In general the symbol of D_j is obtained from the symbol (2) of D by replacing the jth column of D by the column of constants $k_1 \quad , k_n$ Hence if E_j is defined by

$$(22)\quad E_j = \begin{vmatrix} k_1 & a_{11} & & a_{1\,j-1} & a_{1\,j+1} & & a_{1n} \\ k_2 & a_{21} & & a_{2\,j-1} & a_{2\,j+1} & & a_{2n} \\ & & & & & & \\ & & & & & & \\ k_n & a_{n1} & & a_{n\,j-1} & a_{n\,j+1} & & a_{nn} \end{vmatrix}$$

then

$$(23)\qquad D_j = (-1)^{j-1}E_j$$

It is to be noted that (18) and (21) are obtained if j is 2 and 3 in (23) Now for uniformity of notation E_1 is defined to be D_1 Hence (23) is true if $j = 1 \quad n$

Now if equations (23) are used in (14) E becomes

$$(24)\quad k_1D - a_{11}E_1 + a_{12}E_2 - a_{13}E_3 + \quad + (-1)^n a_{1n}E_n$$

In (24) it is to be noted especially that the signs alternate that there are $n + 1$ products and that $D \ E_1 \ E_2 \qquad E_n$ are determinants of order n This suggests that (24) might be the expansion

of a determinant of order $n + 1$, such that its first row is the ordered set $k_1, a_{11}, a_{12}, a_{13}, \cdots, a_{1n}$ and that the minors of these elements are respectively the determinants $D, E_1, E_2, E_3, \cdots, E_n$. It will now be proved that this is indeed the case. Obviously the symbol

$$\begin{vmatrix} k_1 & a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ k_1 & a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ k_2 & a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ k_n & a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

has $k_1, a_{11}, a_{12}, \cdots, a_{1n}$ in its first row, and the minors of these elements are $D, E_1, E_2, \cdots, E_n$. The expansion of this symbol by its first row is

$$(25) \quad (-1)^{1+1}k_1D + (-1)^{1+2}a_{11}E_1 + (-1)^{1+3}a_{12}E_2 + \cdots + (-1)^{1+n+1}a_{1n}E_n.$$

Now the first, second, third, $\cdots$, last terms in (25) equal respectively the first, second, third, $\cdots$, last terms in (24). Therefore the number E in (14) is indeed the determinant whose symbol gave (25) by expansion. Since two rows of this symbol are alike, it follows that E is zero. This completes the proof of (13).

In the same way it is proved that $D_1/D, D_2/D, \cdots, D_n/D$ satisfy each of the equations (1). This result and theorem 1 are combined in theorem 2 and referred to as *Cramer's rule*.

THEOREM 2. *Let $D, D_1, \cdots, D_n$ be defined as in theorem 1. If $D \neq 0$, there is one and only one solution of the equations. This solution is the ordered set of numbers $D_1/D, D_2/D, \cdots, D_n/D$.*

THEOREM 3. *Let $D, D_1, \cdots, D_n$ be defined as in theorem 1. If $D = 0$ and if at least one of $D_1, \cdots, D_n$ is not zero, then the equations are inconsistent.*

PROOF. As in the first part of the proof of theorem 1, if there is a solution of equations (1), (3) are true. Then, by the hypothesis that $D = 0$, it follows that $D_1 = 0, \cdots, D_n = 0$. This contradicts the hypothesis that at least one of $D_1, \cdots, D_n$ is not zero. Therefore there is no solution of (1).

Theorems 2 and 3 give no information about equations (1) if $D = 0$ and $D_1 = 0, \ldots, D_n = 0$. Examples show that if these conditions hold then the equations may be consistent or they may be inconsistent. This was illustrated at the end of chapter 5 if $n = 3$. It will be proved later that a necessary and sufficient condition that equations (1) be consistent is that the rank of the augmented matrix equal the rank of the coefficient matrix.

PROBLEMS

Apply theorems 2 and 3 to the following systems of equations.

1.
$$\begin{aligned} -x + 2y + 10z + 7w &= -28 \\ 2x + y + 20z - w &= -37 \\ 3x + y - 5z + 2w &= 11 \\ x + 7y + 3w &= -2 \end{aligned}$$

2.
$$\begin{aligned} x + 2y - z - 7w &= 6 \\ 3x + y + 4w &= 5 \\ 9x + 2y + z + 5w &= 12 \\ -x + 4y + 2z + 3w &= 5 \end{aligned}$$

3.
$$\begin{aligned} u + v + w - t &= 1 \\ 2u - v - 3w + t &= 2 \\ -u + 2v + 2w - 7t &= -1 \\ -3u + 6v + 4w - 26t &= 1 \end{aligned}$$

4.
$$\begin{aligned} u - 2v + 5w + t &= 2 \\ -u + v + w - 2t &= -1 \\ 4u - v - w + 3t &= 1 \\ 7u - 6v + 8w + 7t &= 0 \end{aligned}$$

5.
$$\begin{aligned} v - s + 2t - 3u &= -9 \\ 2v + t + 3u &= 5 \\ s - 2t + 7u &= 17 \\ -v + 2s + 4u &= 10 \end{aligned}$$

6.
$$\begin{aligned} 2v - 7s + t - 2u &= -22 \\ s - t + 2u &= 4 \\ 3v + 2s - u &= -6 \\ v + 5t - 2u &= -5 \end{aligned}$$

7.
$$\begin{aligned} x + 2y - z + 4u + 7v &= 4 \\ 2x + y + 3u + 5v &= 2 \\ x + 2y + z + 2u + 3v &= 2 \\ -9x + 3y + 3z + 2u - v &= -5 \\ 5x - y - 2z + u &= -4 \end{aligned}$$

8.
$$\begin{aligned}
w + s + t - 2u - v &= 1,\\
2w - s - t \qquad + v &= 0,\\
-3w + 2s \qquad + 2u - v &= 2,\\
w \qquad + 7t - u - v &= 5,\\
-7w + 6s + 2t + 2u - 4v &= 5.
\end{aligned}$$

9.
$$\begin{aligned}
w - 2s + 3t + 2u + 5v &= 1,\\
2w \qquad - t + u - v &= -1,\\
w + 2s + t \qquad + 3v &= 0,\\
4w - s \qquad + u - 2v &= 5,\\
w - 2s + 8t + 3u + 14v &= 1.
\end{aligned}$$

10.
$$\begin{aligned}
5x + y + 2z - u + 4v &= -6,\\
-6x + 4y - 9z - 7u + 2v &= -27,\\
-4x + 3y + 5z - 2u + v &= -5,\\
-x + 2y + z + 3u + 5v &= 21,\\
9x - 3y \qquad + 4u - 4v &= 7.
\end{aligned}$$

2. **Systems of q linear equations in n unknowns.** Consider the system

$$(26) \qquad \begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= k_1,\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= k_2,\\
\cdot \qquad \cdot \qquad & \qquad \cdot \quad \cdot\\
\cdot \qquad \cdot \qquad & \qquad \cdot \quad \cdot\\
\cdot \qquad \cdot \qquad & \qquad \cdot \quad \cdot\\
a_{q1}x_1 + a_{q2}x_2 + \cdots + a_{qn}x_n &= k_q,
\end{aligned}$$

of q linear equations in n unknowns. The *augmented matrix of this system* is the rectangular array

$$(27) \qquad \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & k_1\\
a_{21} & a_{22} & \cdots & a_{2n} & k_2\\
\cdot & \cdot & & \cdot & \cdot\\
\cdot & \cdot & & \cdot & \cdot\\
\cdot & \cdot & & \cdot & \cdot\\
a_{q1} & a_{q2} & \cdots & a_{qn} & k_q
\end{bmatrix}.$$

The *coefficient matrix* of the system is the matrix of q rows and n columns obtained by deleting the last column of (27). The notations a.m. and c.m. will be used for these matrices. The notation (a_{ij}) is also used for the coefficient matrix. It is specifically assumed that there are q equations in the set and that there are n variables in the equations. Hence in each row of (a_{ij}) there is at least one non-zero element, and in each column of (a_{ij}) there is at least one non-zero element.

Let s be a positive integer not larger than q and not larger than $n + 1$ Let s rows and s columns of (27) be selected arbitrarily Then there are s^2 elements of (27) which appear at the intersections of the selected rows with the selected columns These s^2 elements form a square matrix of s rows and columns The determinant of this s rowed square matrix is called an *s-rowed minor* of the a m (27) An s-rowed minor of the c m is obtained by selecting only rows and columns of the c m It is to be noted especially than an s rowed minor of the c m is also an s-rowed minor of the a m, and that, if $s = 1$, then the s-rowed minor is merely an element of the matrix

It was noted that there are many elements of (a_{ij}) which are different from zero Hence there are many one-rowed non-zero minors of (a_{ij}) The *rank r of (a_{ij})* is, by definition, the number of rows in the largest non-zero minor of (a_{ij}) Hence $r \geqq 1$. If each of the two-rowed minors of (a_{ij}) is zero, then the rank r of (a_{ij}) is one But if there is at least one two-rowed non-zero minor of (a_{ij}), then r is greater than one The *rank r_a of the a m* is, by definition, the number of rows in the largest non-zero minor of the a m Since each minor of the c m is also a minor of the a m, the largest non-zero minor of (a_{ij}) is also a non-zero minor of (27) If the largest non-zero minor of (27) is not a minor of (a_{ij}), then $r_a > r$ If the largest non-zero minor of (27) is a minor of (a_{ij}) then $r_a = r$ Hence $r \leqq r_a$

It will be proved now that $r_a = r$ or $r_a = r + 1$ This will be done by proving that, if $r_a \geqq r + 2$, there is a contradiction If $r_a \geqq r + 2$, then $r_a > r + 1$ Hence $r_a - 1 > r$ Since r is the number of rows in the largest non zero minor of (a_{ij}), each $(r_a - 1)$-rowed minor of (a_{ij}) is zero Let M designate an arbitrary, but fixed, one of the largest non-zero minors of (27) Therefore M has r_a rows Let M be expanded by its last column Each minor in this expansion of M is an $(r_a - 1)$-rowed minor of (a_{ij}), and it has already been proved that each $(r_a - 1)$-rowed minor of (a_{ij}) is zero Thus the expansion of M by its last column shows that M is a sum of terms each of which is zero Hence $M = 0$ This contradicts the hypothesis that M is a non-zero minor of (27) The following theorem has been proved

THEOREM 4 *If r is the rank of the coefficient matrix of the equations (26) and if r_a is the rank of the augmented matrix, then $r_a = r$, or $r_a = r + 1$*

The determination of the values of r and r_a, if $q \geqq 4$ and $n + 1 \geqq 4$, would involve evaluating at least one determinant of order four or more. The intricate details of evaluating determinants of large order, which would be involved if r and r_a were evaluated as indicated in their definitions, are avoided by repeated application of the following lemmas 1 and 2. By these lemmas a sequence of matrices can be obtained such that each matrix in the sequence has the same rank that (27) has and the rank of the last matrix in the sequence is determined very simply. Simultaneously a second sequence of matrices is obtained such that each matrix in the second sequence has the same rank that the c.m. of (26) has and the rank of the last matrix is determined simply.

These lemmas will be used now to simplify the determination of r and r_a for the numerical equations

$$\begin{aligned} x - 2y + z - u &= 0, \\ 2x - y - 2z + u &= 0, \\ -x - 4y + 7z - 5u &= 1, \\ 8x - 7y - 4z + u &= 0. \end{aligned}$$

The augmented matrix M_0 of these equations is

$$M_0 = \begin{bmatrix} 1 & -2 & 1 & -1 & 0 \\ 2 & -1 & -2 & 1 & 0 \\ -1 & -4 & 7 & -5 & 1 \\ 8 & -7 & -4 & 1 & 0 \end{bmatrix}.$$

If M_1 designates the matrix which is obtained from M_0 by adding to each element in the second column the product of 2 and the corresponding element in the third column, then

$$M_1 = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 2 & -5 & -2 & 1 & 0 \\ -1 & 10 & 7 & -5 & 1 \\ 8 & -15 & -4 & 1 & 0 \end{bmatrix}.$$

By lemma 1 the rank of M_1 equals the rank of M_0. Later it should be checked that this result is obtained if $q = 4$, $p = 5$,

$s = 2$, $t = 3$, $k = 2$ in lemma 1 If M_2 designates the matrix which is obtained from M_1 by adding to each element in the fourth row the product of -1 and the corresponding element in the second row, then

$$M_2 = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 2 & -5 & -2 & 1 & 0 \\ -1 & 10 & 7 & -5 & 1 \\ 6 & -10 & -2 & 0 & 0 \end{bmatrix}$$

By lemma 2 the rank of M_2 equals the rank of M_1 Later this statement should be checked

It is to be noted especially that multiples of the last column must not be added to the other columns if it is desired to obtain the rank r as well as the rank r_a This is true because the last column of the a m is not a column of the c m or of any matrix obtained from the c m by lemmas 1 and 2 Multiples of other columns may be added to the last column Two or more column transformations may be performed in succession without rewriting the remaining columns if multiples of the same column are used. Also two or more row transformations may be performed similarly However, a row transformation and a column transformation must not be performed in succession without rewriting the remaining elements If M_3 and M_4 are defined by

$$M_3 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 3 & -5 & -1 & 1 & 0 \\ -6 & 10 & 2 & -5 & 1 \\ 6 & -10 & -2 & 0 & 0 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 3 & -5 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & 1 \\ 6 & -10 & -2 & 0 & 0 \end{bmatrix},$$

then it is true that the rank of M_2 equals the rank of M_3, by lemma 1, and that the rank of M_3 equals the rank of M_4, by lemma 2 Now, by inspection there are many two-rowed non-zero minors in M_4, and some of them are in the c m In the first four columns of M_4 each three-rowed minor is zero, but in the upper right-hand corner of M_4 there is a three-rowed non zero minor Hence $r = 2$, and $r_a = 3$

LEMMA 1. *Let C be a matrix having q rows and p columns and symbol (c_{ij}). Let s and t be integers such that $1 \leqq s \leqq p$ and $1 \leqq t \leqq p$, but $s \neq t$. Let k be an arbitrary, fixed number. Let the matrix B be formed from the matrix C as follows: if $j \neq s$, then the jth column of B is precisely the jth column of C; the element in the ith row of the sth column of B is $c_{is} + kc_{it}$ $(i = 1, \cdots, q)$. Then the rank r' of B equals the rank r of C.*

PROOF. It will be proved that $r' \leqq r$. Also it will be proved that $r' \geqq r$. Then it will follow that $r' = r$.

The first part of the proof that $r' \leqq r$ is the proof that, if $r + 1 \leqq p$ and $r + 1 \leqq q$, then each $(r + 1)$-rowed minor of B is zero. Let M be an $(r + 1)$-rowed minor of B. There are three cases: (i) the sth column of B does not occur among the columns of M; (ii) the sth and tth columns of B both occur among the columns of M; (iii) the sth column of B occurs among the columns of M, but the tth column of B does not occur among the columns of M. The proof that $M = 0$ will be made for these three cases separately. If M satisfies condition (i), then M itself is a minor of C, by the method of forming B from C. Also M is $(r + 1)$-rowed, and by the definition of r all $(r + 1)$-rowed minors of C are zero. Hence M is zero. If M satisfies condition (ii), then there is an $(r + 1)$-rowed minor N of C, from which M is obtained by adding to each element of the sth column the product of k and the corresponding element of the tth column. Since M and N are determinants (not matrices), it follows by theorem 13 of chapter 6 that $M = N$. But N is zero, since it is an $(r + 1)$-rowed minor of C and since the rank of C is r. Hence M is zero. Finally, if M satisfies condition (iii), it will be proved that there are two determinants, M_1 and M_2, such that $M = M_1 + kM_2$ and $M_1 = 0$, $M_2 = 0$. By definition, M_1 is obtained from M by replacing the elements $c_{is} + kc_{it}$ of the sth column of M respectively by c_{is}; M_2 is obtained from M by replacing the elements $c_{is} + kc_{it}$ of the sth column of M respectively by c_{it}. Now M_1 is an $(r + 1)$-rowed minor of C, and therefore $M_1 = 0$. Also M_2 is, except perhaps for sign, an $(r + 1)$-rowed minor of C, and therefore $M_2 = 0$. Also $M = M_1 + kM_2$. This completes the proof that each $(r + 1)$-rowed minor of B is zero.

The second part of the proof that $r' \leqq r$ is the proof that, if $r + 2 \leqq p$ and $r + 2 \leqq q$, then each minor of B which has more

than $r + 1$ rows is zero Let M be an $(r + 2)$ rowed minor of B Expansion of M by its first column shows that M is a sum of terms each of which has an $(r + 1)$ rowed minor of B as a factor By the preceding part of this proof each of these $(r + 1)$-rowed minors is zero Therefore M is zero This process can be continued until it has been proved that each minor of B which has more than $r + 1$ rows is zero

The first and second parts of the preceding proof together complete the proof of the fact that $r' \leq r$, because in these parts it has been proved that a minor of B which has more than r rows is zero

It will be proved next that $r \leq r'$ Two auxiliary matrices C_0 and B_0 are used By definition C_0 is the matrix B, and B_0 is the matrix C Therefore the rank r_0 of C_0 is precisely the rank r' of B and the rank r_0' of B_0 is precisely the rank r of C It is to be noted especially that B_0 is formed from C_0 by adding to each element of the sth column of C_0 the product of $-k$ and the corresponding element of the tth column of C_0 The argument which has been applied to B and C and which has led to $r' \leq r$, can be applied to B_0 and C_0 The conclusion is that $r_0' \leq r_0$ Also it has already been noted that $r_0 = r'$ and $r_0' = r$ Therefore $r \leq r'$ This completes the proof of lemma 1

Lemma 2 *If a matrix E is obtained from a matrix C by operating on rows in the same manner as in lemma 1 the matrix B was obtained from the matrix C by operating on columns, then the rank of E equals the rank of C*

Proof By hypothesis the sth row of E is obtained from C by adding to each element of the sth row of C the product of k and the corresponding element of the tth row of C, and each other row of E is precisely the corresponding row of C Let E' be the matrix obtained from E by interchanging rows and columns of E, and let C' be the matrix obtained from C by interchanging rows and columns of C Then by lemma 1 applied to E' and C', it is true that the rank of E' equals the rank of C' Also a minor in E is zero if and only if its corresponding minor in E' is zero The same statement is true of C and C' Hence the rank of E equals the rank of E', and the rank of C equals the rank of C' Therefore the rank of E equals the rank of C This completes proof of lemma 2

PROBLEMS

Find r and r_a for each of the following systems of equations.

1. $$\begin{aligned} x + 7y + 2z + u &= 1, \\ -x - y + z - 4u &= 0, \\ 2x + 2y - 2z + 5u &= 4, \\ 4x - 2y - 7z + 10u &= 11. \end{aligned}$$

2. $$\begin{aligned} 5x + 2y - z + u &= 0, \\ x - 4y + 7z - 4u &= -1, \\ 2x - y + z - u &= 7, \\ 9x + y + 4z - u &= -8. \end{aligned}$$

3. $$\begin{aligned} v + 2s - t + 5u &= -1, \\ 2v + 11s - 7t + 26u &= 1, \\ 3v - s + 2t - u &= 3, \\ 5v + 3s + 9u &= 1. \end{aligned}$$

4. $$\begin{aligned} -2v + s - 7t - u &= 2, \\ -8v + 13s - 23t - 5u &= 4, \\ 3v + 3s + 13t + u &= 0, \\ v - 5s + t + u &= 1. \end{aligned}$$

5. $$\begin{aligned} x - y + 4z + 2t &= 0, \\ -11x - 8y + z - 9t &= 9, \\ 7x + 5y + z + t &= -2, \\ 22x + 19y - 2z - 8t &= -1, \\ 2x + 3y - z - 9t &= 5. \end{aligned}$$

6. $$\begin{aligned} 5x + y - z + 2t &= 1, \\ 14x + 20y + 2z + 24t &= -1, \\ 12x - y - 3z - 5t &= 2, \\ 3x + 2y - t &= -5, \\ 4x + 7y + z + 7t &= 0 \end{aligned}$$

7. $$\begin{aligned} x - y + 2z + u + 3v &= 1, \\ -5x - 2y + 21u + 12v &= 3, \\ -x + y + 7u + 2v &= -1, \\ 7x + 4y + z + 3u &= 2, \\ 3x + y - z + 9u + 5v &= 0. \end{aligned}$$

8. $$\begin{aligned} 2x - y + 3z - u + v &= 2, \\ 2y + z + 2u + 4v &= 4, \\ -4x + 5y - 2z + 5u &= 3, \\ -x + 4y + 5z - 2v &= 1, \\ x - 3y - 2z + u &= 7. \end{aligned}$$

9
$$\begin{aligned}
2y + s + 5t + 2v - w &= 1\\
-4y - 4s + 8t + 10v - 5w &= 2\\
8y + 5s + 13t + 10v - 2w &= 1\\
8y + 6s + 2t - 6v + 3w &= 0\\
5y + 4s \qquad - v + 2w &= -1\\
-y - s + 3t + 7v - 2w &= 0
\end{aligned}$$

10
$$\begin{aligned}
3y - s \qquad - 5v + w &= 1\\
-y + s - 2t + v \qquad &= -1\\
2y - 4s - 2t - 12v + w &= 3\\
-2y + 2s + t + 3v - w &= 0\\
y - 5s + 2t - 9v \qquad &= 4\\
5y + s - t \qquad + 2w &= -2
\end{aligned}$$

There are two ways in which the proofs of the fundamental theorems 5 and 6 will be simplified These ways will be illustrated by means of the system of equations

$$(28)\qquad \begin{aligned}
7x - 2y + 59z + 11u + 15v &= 70\\
2x + y + 9z + 3u - 3v &= 12,\\
x + 6y - 23z + u - 27v &= -22,\\
x + 2y - 3z + u - 7v &= -6,\\
3x - y + 26z + 4u + 10v &= 22,\\
3x - 4y + 41z + 5u + 21v &= 46
\end{aligned}$$

The a m of equations (28) is

$$(29)\qquad \begin{bmatrix}
7 & -2 & 59 & 11 & 15 & 70\\
2 & 1 & 9 & 3 & -3 & 12\\
1 & 6 & -23 & 1 & -27 & -22\\
1 & 2 & -3 & 1 & -7 & -6\\
3 & -1 & 26 & 4 & 10 & 22\\
3 & -4 & 41 & 5 & 21 & 46
\end{bmatrix}$$

By lemmas 1 and 2 it is found that $r = 3$ and $r_a = 3$ The last step in this process indicates that the coefficients of y, z u in the second, fourth and fifth equations in (28) form a non zero third order minor of the c m of (28) This minor will be designated by M Hence M is

$$(30)\qquad \begin{vmatrix}
1 & 9 & 3\\
2 & -3 & 1\\
-1 & 26 & 4
\end{vmatrix}$$

Now, if the equations are rearranged as in

$$
(31)\qquad
\begin{aligned}
2x + y + 9z + 3u - 3v &= 12,\\
x + 2y - 3z + u - 7v &= -6,\\
3x - y + 26z + 4u + 10v &= 22,\\
7x - 2y + 59z + 11u + 15v &= 70,\\
x + 6y - 23z + u - 27v &= -22,\\
3x - 4y + 41z + 5u + 21v &= 46,
\end{aligned}
$$

then M appears in the first three rows of the a.m.

$$
\begin{bmatrix}
2 & 1 & 9 & 3 & -3 & 12\\
1 & 2 & -3 & 1 & -7 & -6\\
3 & -1 & 26 & 4 & 10 & 22\\
7 & -2 & 59 & 11 & 15 & 70\\
1 & 6 & -23 & 1 & -27 & -22\\
3 & -4 & 41 & 5 & 21 & 46
\end{bmatrix}.
$$

A set of numbers which constitute a solution of (28) is a set of numbers which constitute a solution of (31). Conversely, a solution of (31) is a solution of (28). Therefore (31) and (28) are equivalent. This illustrates the general fact that, if the equations in a system are rearranged in any desired order, then the new system is equivalent to the original system. It is also true that the ranks of the coefficient matrices of the two systems are equal and that the ranks of the augmented matrices of the two systems are equal.

The second way in which proofs will be simplified will be illustrated next. If the variables in (31) are renamed by writing

$$
(32)\qquad X = y,\quad Y = z,\quad Z = u,\quad U = x,\quad V = v,
$$

then (31) become

$$
(33)\qquad
\begin{aligned}
X + 9Y + 3Z + 2U - 3V &= 12,\\
2X - 3Y + Z + U - 7V &= -6,\\
-X + 26Y + 4Z + 3U + 10V &= 22,\\
-2X + 59Y + 11Z + 7U + 15V &= 70,\\
6X - 23Y + Z + U - 27V &= -22,\\
-4X + 41Y + 5Z + 3U + 21V &= 46.
\end{aligned}
$$

It is especially to be noted that (32) can be regarded as a reordering of the variables in (31) which results in the system (33). The a.m. of (33) is

$$(34)\qquad \begin{bmatrix} 1 & 9 & 3 & 2 & -3 & 12 \\ 2 & -3 & 1 & 1 & -7 & -6 \\ -1 & 26 & 4 & 3 & 10 & 22 \\ -2 & 59 & 11 & 7 & 15 & 70 \\ 6 & -23 & 1 & 1 & -27 & -22 \\ -4 & 41 & 5 & 3 & 21 & 46 \end{bmatrix}$$

In this matrix M appears in the upper left-hand corner. Now by (32), a solution of (31) after being reordered is a solution of (33) and a solution of (33) after being reordered is a solution of (31). Hence (31) and (33) are equivalent. This illustrates the general fact that if the variables in a system of equations are reordered in any desired way, then the new system is equivalent to the original system. It is also true that the ranks of the coefficient matrices of the two systems are equal and that the ranks of the augmented matrices of the two systems are equal.

THEOREM 5. *If the rank of the augmented matrix of a set of linear equations is not equal to the rank of the coefficient matrix of the equations, then the equations are inconsistent.*

PROOF. It will be proved that if there is a solution, then there is a contradiction. Let r be the rank of the c.m. of the equations. By theorem 4 and the hypothesis that the rank of the a.m. is not equal to the rank of the c.m., there is an $(r + 1)$-rowed non-zero minor in the a.m. This minor will be designated by M. Now the elements of a row of M form an ordered sub-set of the elements of a unique row of the a.m. It will be said that the row of M determines this row of the a.m. and the corresponding equation in the system. Again, the elements of a column of M form an ordered sub-set of the elements of a unique column of the a.m. The column of M is said to determine this column of the a.m. By its definition M is non-zero and has $r + 1$ columns. Also the largest non-zero minor of the c.m. has r columns. Hence the last column of M determines the last column of the a.m. Each other column of M determines a variable in the system. Thus M determines $r + 1$ equations and r variables.

Now the original equations can be rearranged so that the equations determined by M, their relative positions preserved, are the first $r + 1$ equations in the rearranged set. Then the variables

in the equations of this rearranged set can be reordered so that the first r columns determined by M, their relative positions preserved, are in the upper left-hand corner of the a.m. of the final set of equations. The last column of M is precisely the first $r+1$ elements of the last column of the a.m. of the final set of equations.

The rearranged equations are given the notation (26). Hence M is non-zero and has the symbol

$$\begin{vmatrix} a_{11} & \cdots & a_{1r} & k_1 \\ a_{21} & \cdots & a_{2r} & k_2 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_{r1} & \cdots & a_{rr} & k_r \\ a_{r+1,1} & \cdots & a_{r+1,r} & k_{r+1} \end{vmatrix}.$$

Also, $x_1, \cdots, x_n$ satisfy (26) because the original equations have a solution by hypothesis and the variables $x_1, \cdots, x_n$ are merely the original variables reordered. Therefore M has the symbol

$$(35) \quad \begin{vmatrix} a_{11} & \cdots & a_{1r} & (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) \\ a_{21} & \cdots & a_{2r} & (a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n) \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_{r1} & \cdots & a_{rr} & (a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rn}x_n) \\ a_{r+1,1} & \cdots & a_{r+1,r} & (a_{r+1,1}x_1 + a_{r+1,2}x_2 + \cdots + a_{r+1,n}x_n) \end{vmatrix}.$$

Now, by theorem 12 of chapter 6, the determinant (35) equals the sum

$$\begin{vmatrix} a_{11} & \cdots & a_{1r} & a_{11}x_1 \\ a_{21} & \cdots & a_{2r} & a_{21}x_1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_{r1} & \cdots & a_{rr} & a_{r1}x_1 \\ a_{r+1,1} & \cdots & a_{r+1,r} & a_{r+1,1}x_1 \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & \cdots & a_{1r} & (a_{12}x_2 + \cdots + a_{1n}x_n) \\ a_{21} & \cdots & a_{2r} & (a_{22}x_2 + \cdots + a_{2n}x_n) \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_{r1} & \cdots & a_{rr} & (a_{r2}x_2 + \cdots + a_{rn}x_n) \\ a_{r+1,1} & \cdots & a_{r+1,r} & (a_{r+1,2}x_2 + \cdots + a_{r+1,n}x_n) \end{vmatrix}.$$

The second determinant can also be written as a sum. Repetition of this process shows that (35) is a sum of n determinants. The factor x_1 is in the last column of the first of these n determinants, the factor x_2 is in the last column of the second of these n determinants, . . . , the factor x_n is in the last column of the last of these n determinants. Thus if

$$(36)\quad M_j = \begin{vmatrix} a_{11} & \cdots & a_{1r} & a_{1j} \\ a_{21} & \cdots & a_{2r} & a_{2j} \\ \vdots & & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{rj} \\ a_{r+1\,1} & \cdots & a_{r+1\,r} & a_{r+1\,j} \end{vmatrix} \qquad (j = 1, \ldots, n),$$

then $M = M_1x_1 + M_2x_2 + \cdots + M_nx_n$.

It will be proved next that $M_1 = 0, \ldots, M_n = 0$. If $j > r$, then by (36) M_j is an $(r + 1)$ rowed minor of the c.m. of (26). But the largest non zero minor of the c.m. has r rows. Hence $M_j = 0$ if $j > r$. Again if $j \leq r$ then by (36) M_j has two columns alike and hence $M_j = 0$. Since M is a sum of n terms each of which is zero, $M = 0$. This contradicts the hypothesis that M is a non zero minor of the a.m.

PROBLEMS

1. Apply theorem 5 to those systems of equations in the set of problems on page 193 to which it is applicable. Do the same for theorem 2. Why is neither theorem 5 nor theorem 2 applicable to the remaining problems in the set?

2. Show that theorem 3 of chapter 5 is a special case of theorem 5 and that theorem 1 and theorem 2 of chapter 5 constitute a special case of theorem 2.

Apply theorem 5 to those of the following systems of equations to which it is applicable.

3.
$$\begin{aligned} x + 2y - z \phantom{{}+4u} + 7v &= 1 \\ -x + 3y + z + 4u - v &= 0 \\ 4x + y \phantom{{}+z} - 2u \phantom{{}+7v} &= -1 \\ -2y + z - u + 3v &= 5 \\ -2x - 4y - z - 3u + 11v &= 2 \end{aligned}$$

4.
$$\begin{aligned} x + 3y - 2z - u + 2v &= 4 \\ 5x + 6y \phantom{{}-2z} - u + 7v &= 10 \\ 2x + y + 4z \phantom{{}-u} + 3v &= 1 \\ -x + 2y - z + 2u - v &= 0 \\ -y + z - 3u + v &= -1 \end{aligned}$$

5.
$$\begin{aligned}
x + 5y - 4z + 2u - 10v &= 0,\\
3y + z - u + 7v &= 2,\\
-x + 5y - 11z + 2u - 10v &= -3,\\
-x - 12y + 5z - 3u + 15v &= 4,\\
4x - y + 2z + u - 3v &= 1,\\
3x + y + 5z + 2v &= -1.
\end{aligned}$$

6.
$$\begin{aligned}
4x - 7y + 4z - 6u - 6v &= -10,\\
2x - y - 4u + v &= -3,\\
x - z + 2u - v &= -2,\\
-x + 4y - 3z + 5v &= 7,\\
x - 2y + z - 2u - 2v &= 0,\\
4x - 5y + 2z - 2u - 5v &= -12.
\end{aligned}$$

7.
$$\begin{aligned}
5x + 2y - z + u + 5v &= 0,\\
3x - 3y - 5z - 6u + 5v &= -1,\\
7x - 5y - 3z - 13u + 2v &= 4,\\
2x + y + 2z - v &= 2,\\
3x + 9y + z + 15u + 8v &= -4,\\
4y + 2z + 7u + v &= -1.
\end{aligned}$$

8.
$$\begin{aligned}
x + y + 5z - u &= 1,\\
-2x - 4y + u + v &= -3,\\
4x - 6y - 3z - u + 2v &= 2,\\
2x - 12y + 7z - u + 4v &= 0,\\
17x + 3y + 9z - 8u + v &= -1,\\
7x - y + 2z - 3u + v &= 4.
\end{aligned}$$

The system (28) of equations will now be solved by methods which will illustrate all the ideas and notations in the proof of the general theorem that, if the rank of the augmented matrix of a set of linear equations equals the rank of the coefficient matrix, then there is at least one solution of the equations. It has already been proved that the system (28) is equivalent to the system (33).

The first three equations of the system (33) will now be solved, by Cramer's rule, for X, Y, Z in terms of U and V. First these three equations are written in the form

$$\begin{aligned}
X + 9Y + 3Z &= 12 - 2U + 3V,\\
2X - 3Y + Z &= -6 - U + 7V,\\
-X + 26Y + 4Z &= 22 - 3U - 10V.
\end{aligned}\tag{37}$$

If the notation

$$\begin{aligned}
K_1 &= 12 - 2U + 3V,\\
K_2 &= -6 - U + 7V,\\
K_3 &= 22 - 3U - 10V,
\end{aligned}\tag{38}$$

is used then (37) become

$$
(39)\qquad
\begin{aligned}
X + 9Y + 3Z &= K_1\\
2X - 3Y + Z &= K_2\\
-X + 26Y + 4Z &= K_3
\end{aligned}
$$

Now let D_1 designate the determinant

$$
(40)\qquad
\begin{vmatrix} K_1 & 9 & 3\\ K_2 & -3 & 1\\ K_3 & 26 & 4 \end{vmatrix}
$$

Hence by Cramer's rule and (30) $X = D_1/M$. Similarly if D_2 and D_3 are defined by

$$
(41)\quad D_2 = \begin{vmatrix} 1 & K_1 & 3\\ 2 & K_2 & 1\\ -1 & K_3 & 4 \end{vmatrix} \qquad D_3 = \begin{vmatrix} 1 & 9 & K_1\\ 2 & -3 & K_2\\ -1 & 26 & K_3 \end{vmatrix}
$$

then $Y = D_2/M$ and $Z = D_3/M$. Expansion of D_1 by its first column shows that $D_1 = -38K_1 + 42K_2 + 18K_3$. Hence by (38)

$$
(42)\qquad D_1 = -312 - 20U
$$

In the same way it is proved that

$$
(43)\qquad
\begin{aligned}
D_2 &= -40 - 4U - 28V\\
D_3 &= \ \ 336 \qquad\quad + 112V
\end{aligned}
$$

By (30) $M = 28$. Therefore the solution of the first three of equations (33) for X, Y, Z in terms of U and V is

$$
(44)\qquad
\begin{aligned}
X &= (-312 - 20U \qquad\qquad)/28\\
Y &= (\ -40 - \ 4U - \ 28V)/28\\
Z &= (\ \ 336 \qquad\qquad + 112V)/28
\end{aligned}
$$

It will now be proved that *the expressions (44) which have been obtained from the first three equations in the system (33) satisfy the remaining equations in system (33)*. Fractions are avoided if (33_4) is multiplied by 28 before the expressions (44) are used. Thus (33_4) is equivalent to

$$
(45)\quad -2\cdot 28X + 59\cdot 28Y + 11\cdot 28Z + 196U + 420V = 1960
$$

Therefore (44) satisfy (33_4) if and only if

$$(46)\quad -2(-312-20U)+59(-40-4U-28V)$$
$$+11(336+112V)+196U+420V=1960.$$

In this expression the coefficient of U is zero. Also the coefficient of V is zero. Hence (44) satisfy (33_4) if and only if

$$(47)\qquad -2(-312)+59(-40)+11\cdot 336=1960.$$

Now (47) is true regardless of the values of U and V in (44). Therefore for all values of U and V (44) satisfy (33_4). This is the meaning of the statement that *(44) satisfy (33_4) identically in U and V.* Similarly it is proved that the expressions (44) satisfy (33_5) and (33_6) identically in U and V.

PROBLEMS

In the two preceding lists of problems solve the systems of equations which have $r = r_a$ by the method of the preceding illustration. In each problem verify that the expressions obtained satisfy the other equations in the system identically in the transposed variables.

Another method of showing that the expressions (44) satisfy the remaining equations in (33) will now be explained because it illustrates the method used in the general proofs. If the functions f_1, f_2, f_3, f_4 are defined by

$$(48)\quad \begin{aligned} f_1 &= X+9Y+3Z+2U-3V-12,\\ f_2 &= 2X-3Y+Z+U-7V+6,\\ f_3 &= -X+26Y+4Z+3U+10V-22,\\ f_4 &= -2X+59Y+11Z+7U+15V-70,\end{aligned}$$

then the first four of equations (33) become $f_1=0$, $f_2=0$, $f_3=0$, $f_4=0$. Also, if the first function is multiplied by 3, the second by -2, and the third by 1, and if these results are added, it is found that the fourth function is obtained. Therefore

$$(49)\qquad f_4 \equiv 3f_1-2f_2+f_3.$$

By (49) values of X, Y, Z, U, V which make each of the functions f_1, f_2, and f_3 zero are values which also make f_4 zero. This means that a solution of $f_1=0$, $f_2=0$, $f_3=0$ is a solution of $f_4=0$. Hence the general solution (44) of (33_1), (33_2), and (33_3) satisfies (33_4).

In the same way it is proved that the expressions (44) satisfy (33_5) and (33_6) Thus, if f_5 and f_6 designate the functions $6X - 23Y + Z + U - 27V + 22$ and $-4X + 41Y + 5Z + 3U + 21V - 46$ respectively, then

$$(50) \qquad f_5 \equiv 3f_1 - 2f_2 + f_3, \quad \text{and} \quad f_6 \equiv f_1 - 2f_2 + f_3$$

This method of proving that the general solution (44) of the first three equations in (33) satisfies the remaining equations in (33) is called the method of linear dependence This is done because the existence of an identity such as (49) is precisely the meaning of the statement that f_4 *is a linear combination of* f_1, f_2 *and* f_3 It is also said that the equation (33_4) is linearly dependent on the three equations (33_1), (33_2), (33_3) In the same way (50_1) shows that (33_5) is linearly dependent on (33_1), (33_2), (33_3), and (50_2) shows that (33_6) is linearly dependent on (33_1), (33_2), (33_3)

In the preceding proof (49) was verified However, verification does not illustrate the ideas and notations in the proof of the general theorem A proof of (49) which illustrates these ideas will be given now Let T designate the fourth-order determinant formed from (48) by the coefficients of X, Y, Z and the column of constants Then

$$(51) \qquad T = \begin{vmatrix} 1 & 9 & 3 & -12 \\ 2 & -3 & 1 & 6 \\ -1 & 26 & 4 & -22 \\ -2 & 59 & 11 & -70 \end{vmatrix}$$

By (30) the minor of the -70 in the lower right hand corner of T is M Let M_1 M_2 M_3 designate the minors of the other elements in the last column of (51) It will be proved by a general method that

$$(52) \qquad M_1f_1 - M_2f_2 + M_3f_3 - Mf_4 = 0$$

However before this is done it will be verified that (52) implies (49) By the definitions

$$(53) \quad M_1 = \begin{vmatrix} 2 & -3 & 1 \\ -1 & 26 & 4 \\ -2 & 59 & 11 \end{vmatrix}, \quad M_2 = \begin{vmatrix} 1 & 9 & 3 \\ -1 & 26 & 4 \\ -2 & 59 & 11 \end{vmatrix},$$

$$M_3 = \begin{vmatrix} 1 & 9 & 3 \\ 2 & -3 & 1 \\ -2 & 59 & 11 \end{vmatrix}$$

Therefore

$$(54)\qquad M_1 = 84,\quad M_2 = 56,\quad M_3 = 28,\quad M = 28.$$

If these values are used in (52) and the result is divided by 28, the identity (49) is obtained.

It will be proved now that (52) is true. The left-hand side of (52) suggests that (48_1) be multiplied by M_1, (48_2) by $-M_2$, (48_3) by M_3, (48_4) by $-M$, and the results added. The function so obtained will not be displayed because of its length. In it the coefficients of X, Y, Z, U, V are

$$(55)\qquad M_1\cdot 1 - M_2\cdot 2 + M_3(-1) - M(-2),$$

$$(56)\qquad M_1\cdot 9 - M_2(-3) + M_3\cdot 26 - M\cdot 59,$$

$$(57)\qquad M_1\cdot 3 - M_2\cdot 1 + M_3\cdot 4 - M\cdot 11,$$

$$(58)\qquad M_1\cdot 2 - M_2\cdot 1 + M_3\cdot 3 - M\cdot 7,$$

$$(59)\qquad M_1(-3) - M_2(-7) + M_3\cdot 10 - M\cdot 15,$$

respectively. The constant term is

$$(60)\qquad M_1(-12) - M_2\cdot 6 + M_3(-22) - M(-70).$$

Therefore, (52) is true if and only if each of the numbers (55), (56), (57), (58), (59), (60) is indeed zero.

To prove that the number (60) is zero the expansion of (51) by its last column is used. Thus

$$(61)\qquad T = -(-12)M_1 + 6M_2 - (-22)M_3 + (-70)M.$$

On the other hand, if the last column of T is multiplied by -1, the resulting determinant is $-T$. Also this determinant is obviously a four-rowed minor of (34). Since $r_a = 3$, it follows that $-T = 0$. Hence $T = 0$, and (61) becomes

$$(62)\qquad 0 = -(-12)M_1 + 6M_2 - (-22)M_3 + (-70)M.$$

Multiplication of (62) by -1 shows that the number (60) is zero.

To prove that the number (59) is zero let T_5 designate the determinant formed by replacing the last column of T by the coefficients of the fifth variable V in (48). Then

$$T_5 = \begin{vmatrix} 1 & 9 & 3 & -3 \\ 2 & -3 & 1 & -7 \\ -1 & 26 & 4 & 10 \\ -2 & 59 & 11 & 15 \end{vmatrix}.$$

Expansion of T_5 by its last column gives

$$T_5 = -(-3)M_1 + (-7)M_2 - M_3\ 10 + M\ 15$$

Also, $T_5 = 0$, since T_5 is a four-rowed minor of (34) Hence

$$(63) \qquad 0 = -(-3)M_1 + (-7)M_2 - M_3\ 10 + M\ 15$$

Multiplication of (63) by -1 shows that (59) is zero A similar proof shows that (58) is zero

To prove that (57) is zero let T_3 designate the determinant obtained by replacing the last column of (51) by the coefficients of Z in (48) Then

$$T_3 = \begin{vmatrix} 1 & 9 & 3 & 3 \\ 2 & -3 & 1 & 1 \\ -1 & 26 & 4 & 4 \\ 2 & 59 & 11 & 11 \end{vmatrix}$$

Now expansion of T_3 by its last column gives

$$T_3 = -3M_1 + M_2 - 4M_3 + 11M$$

Also $T_3 = 0$, since it has two columns alike Hence

$$(64) \qquad 0 = -3M_1 + M_2 - 4M_3 + 11M$$

Multiplication of (64) by -1 shows that (57) is zero A similar proof shows that (55) and (56) are zero This completes the proof of (52)

The general rule is that the first r rows of a determinant of order $r + 1$ are formed from the columns of M and the column of constants in the equations from which the solution was obtained, and that the last row consists of the corresponding coefficients in the equation whose dependence is being exhibited The coefficients in the linear dependence are then the signed minors of the elements of the last column of this $(r + 1)$ rowed determinant For example the coefficients in (50_1) are obtained by applying the preceding method to the first second third, and fifth of equations (33) and the coefficients in (50_2) from the first, second, third, and sixth of equations (33)

PROBLEMS

1 By the methods just illustrated find the linear dependence which was exhibited in (61) of chapter 5

2 Do the same for each of the remaining equations in each problem on page 201

It will now be proved in general that, if the rank of the augmented matrix of a set of linear equations equals the rank of the coefficient matrix of the set, then there is at least one solution of the equations. Let r be the rank of the c.m. of the equations. Then there is an r-rowed non-zero minor of the c.m. This minor will be designated by M. Now M determines r equations and r variables. Let the equations be rearranged and the variables be reordered so that the rows and columns of M form the upper left-hand corner of the c.m. of the new set. Let them have the notation (26). Then M is non-zero and has the symbol

$$(65)\qquad \begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{r1} & \cdots & a_{rr} \end{vmatrix}.$$

Now, if $r < n$, then the first r equations in (26) can be written

$$(66)\qquad \begin{array}{l} a_{11}x_1 + \cdots + a_{1r}x_r = k_1 - (a_{1,r+1}x_{r+1} + \cdots + a_{1n}x_n), \\ \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\ \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\ \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\ a_{r1}x_1 + \cdots + a_{rr}x_r = k_r - (a_{r,r+1}x_{r+1} + \cdots + a_{rn}x_n). \end{array}$$

It will be convenient to use the notation

$$(67)\qquad k'_i = k_i - (a_{i,r+1}x_{r+1} + \cdots + a_{in}x_n) \quad (i = 1, \cdots, r < n).$$

Hence, if $r < n$, the first r of equations (26) become

$$(68)\qquad \begin{array}{l} a_{11}x_1 + \cdots + a_{1r}x_r = k'_1, \\ \cdot \qquad \cdot \qquad \cdot \\ \cdot \qquad \cdot \qquad \cdot \\ \cdot \qquad \cdot \qquad \cdot \\ a_{r1}x_1 + \cdots + a_{rr}x_r = k'_r. \end{array}$$

To avoid treating the cases $r < n$ and $r = n$ separately the notation

$$(69)\qquad k'_i = k_i \quad (i = 1, \cdots, n)$$

will be used if $r = n$. Hence, if $r = n$, the first r equations in (26) can also be written as (68).

The determinant of the coefficients of the r variables in the r equations (68) is M. Also M is non-zero. Hence (68) can be

solved by Cramer's rule Here r, M, k'_1, $\ldots$, k'_r replace n, D, k_1, $\ldots$, k_n of theorem 2 Let D_1 be the determinant

$$(70) \qquad \begin{vmatrix} k'_1 & a_{12} & \cdots & a_{1r} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ k'_r & a_{r2} & \cdots & a_{rr} \end{vmatrix}$$

Then $x_1 = D_1/M$ Now, if $r = n$, it is true by (69) that D_1 here is precisely D_1 of theorem 2 Therefore each x_i has the value given by theorem 2 Next, let $r < n$ The expansion of (70) by minors of its first column gives

$$(71) \qquad D_1 = \sum_{i=1}^{r} (-1)^{i+1} k'_i A_{i1}$$

Hence by (67)

$$(72) \quad D_1 = \sum_{i=1}^{r} (-1)^{i+1} A_{i1}[k_i - (a_{i\,r+1}x_{r+1} + \cdots + a_{in}x_n)]$$

The coefficient of x_{r+1} in (72) is

$$(73) \quad -[(-1)^{1+1}A_{11}a_{1\,r+1} + (-1)^{2+1}A_{21}a_{2\,r+1} + \cdots + (-1)^{r+1}A_{r1}a_{r\,r+1}]$$

Let this number be designated by $b_{1\,r+1}$ In general, let b_{1j} designate the coefficient of x_j in (72) if $j = r+1, \ldots, n$ Also let $b_{1\,0}$ designate the constant in D_1 Then (72) becomes

$$(74) \qquad D_1 = b_{1\,0} + b_{1\,r+1}x_{r+1} + \cdots + b_{1n}x_n$$

Therefore

$$(75) \qquad x_1 = \frac{b_{1\,0}}{M} + \frac{b_{1\,r+1}}{M}x_{r+1} + \cdots + \frac{b_{1n}}{M}x_n$$

It is to be noted especially that, if $r < n$, then the solution by Cramer's rule determines x_1 as a unique linear function (75) of the transposed variables x_{r+1}, $\ldots$, x_n This function is a linear homogeneous function of x_{r+1}, $\ldots$, x_n if and only if the constant term $b_{1\,0}/M$ is zero

In the same way it is proved that, if D_j is obtained from M by replacing the jth column of M by k'_1, $\ldots$, k'_r, then $x_j = D_j/M$ Expansion of D_j by minors of its jth column shows that x_j is a

unique linear function of the transposed variables $x_{r+1}, \cdots, x_n$. This function is a linear homogeneous function of these variables if and only if its constant term is zero.

If values $c_{r+1}, \cdots, c_n$ are assigned respectively to $x_{r+1}, \cdots, x_n$, then by (75) a unique value, called c_1, is determined for x_1. Similarly there is determined a unique value for x_j ($j = 1, \cdots, r$). Thus a numerical solution of the first r equations of (26) is obtained by assigning arbitrary values to the transposed variables.

PROBLEMS

From the results obtained for the problems on page 201 obtain three numerical solutions for each system.

It will now be proved that the expressions for $x_1, \cdots, x_r$ in terms of $x_{r+1}, \cdots, x_n$, which have been obtained from the first r equations, satisfy each of the remaining equations identically in $x_{r+1}, \cdots, x_n$. This means that each set of values, which consists of values assigned to $x_{r+1}, \cdots, x_n$ and the values of $x_1, \cdots, x_r$ obtained from them, satisfies the remaining equations. If $r = q$, there are no remaining equations. If $r < q$, let s be an arbitrary integer such that $r < s \leqq q$. It will be proved that $D_1/M, \cdots, D_r/M$ satisfy the sth equation for all values of $x_{r+1}, \cdots, x_n$.

If the functions $f_1, \cdots, f_q$ are defined by

(76) $$f_i = a_{i1}x_1 + \cdots + a_{in}x_n - k_i \quad (i = 1, \cdots, q),$$

then equations (26) can be written in the form $f_1 = 0, \cdots, f_q = 0$. Let T designate the $(r + 1)$-rowed determinant formed from the first r and the sth of these equations by the coefficients of $x_1, \cdots, x_r$ and the constant terms. Thus

(77) $$T = \begin{vmatrix} a_{11} & \cdots & a_{1r} & -k_1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_{r1} & \cdots & a_{rr} & -k_r \\ a_{s1} & \cdots & a_{sr} & -k_s \end{vmatrix}.$$

Then M is the minor of the element $-k_s$ in the lower right-hand corner of T. Let $M_1, \cdots, M_r$ designate the minors of $-k_1, \cdots, -k_r$ respectively. It will be proved that

(78) $$Mf_s \equiv (-1)^{r+1}M_1f_1 + (-1)^{r+2}M_2f_2 + \cdots + (-1)^{r+r}M_rf_r.$$

It will follow then that any values of $x_1, \ldots, x_r$ which make each of the functions $f_1, \ldots, f_r$ zero also make Mf_s zero, and hence make f_s zero. That is the expressions for $x_1, \ldots, x_r$ which by the manner of their derivation satisfy the first r of equations (26) for all values of $x_{r+1}, \ldots, x_n$ will also satisfy the sth equation in (26) for all values of $x_{r+1}, \ldots, x_n$.

Now (78) is equivalent to

$$(-1)^{r+1}M_1f_1 + (-1)^{r+2}M_2f_2 + \cdots + (-1)^{r+r}M_rf_r + (-1)^{r+r+1}Mf_s \equiv 0$$

and hence to

$$(79) \quad (-1)^{r+2}M_1f_1 + (-1)^{r+3}M_2f_2 + \cdots + (-1)^{2r+1}M_rf_r + (-1)^{2r+2}Mf_s \equiv 0$$

The left-hand side of this equation suggests that the first equation in (76) be multiplied by $(-1)^{r+2}M_1$, the second by $(-1)^{r+3}M_2, \ldots$, the rth by $(-1)^{2r+1}M_r$, the sth by $(-1)^{2r+2}M$, and the results added. In the final equation the coefficient of x_j is precisely

$$(80) \quad (-1)^{r+2}M_1a_{1j} + \cdots + (-1)^{2r+1}M_ra_{rj} + (-1)^{2r+2}Ma_{sj} \quad (j = 1, \ldots, n)$$

The constant term is

$$(81) \quad (-1)^{r+2}M_1(-k_1) + \cdots + (-1)^{2r+1}M_r(-k_r) + (-1)^{2r+2}M(-k_s)$$

It will now be proved that for each value of j the number (80) is zero and that the number (81) is zero. This will prove (79). The expansion of (77) by its last column gives

$$(82) \quad (-1)^{1+r+1}(-k_1)M_1 + \cdots + (-1)^{r+r+1}(-k_r)M_r + (-1)^{r+1+r+1}(-k_s)M = T$$

On the other hand, if the last column of T is multiplied by -1 the resulting determinant is an $(r+1)$ rowed minor of the a.m. of (26). Hence $-T = 0$. Hence $T = 0$ and (82) becomes

$$(83) \quad (-1)^{r+2}M_1(-k_1) + \cdots + (-1)^{2r+1}M_r(-k_r) + (-1)^{2r+2}M(-k_s) = 0$$

This proves that the number (81) is zero. To prove that for each value of j the number (80) is zero T_j is defined as the determinant obtained by replacing the last column of T by the coefficients of x_j. If $j \leqq r$, then T is zero because it has two columns alike. If $r < j \leqq n$, then T_j is an $(r + 1)$-rowed minor of the c.m. and hence is zero. On the other hand, the expansion of T_j by its last column gives

$$(-1)^{1+r+1}a_{1j}M_1 + \cdots + (-1)^{r+r+1}a_{rj}M_r \\ + (-1)^{r+1+r+1}a_{sj}M = T_j.$$

Therefore

$$(84) \quad (-1)^{r+2}M_1a_{1j} + \cdots + (-1)^{2r+1}M_ra_{rj} \\ + (-1)^{2r+2}Ma_{sj} = 0 \quad (j = 1, \cdots, n).$$

This proves that for each value of j the number (80) is zero. The proof that (79) is true has been completed.

These facts and theorem 5 complete the proof of the following fundamental theorem 6.

THEOREM 6. *A system of q linear equations in n variables is consistent if and only if the rank of the augmented matrix of the equations equals the rank r of the coefficient matrix of the equations. If these ranks are equal, then there is a subset of r equations and a subset of r variables such that the equations in the subset can be solved for these r variables. The solution expresses each of these r variables as a unique linear function of the remaining $n - r$ variables. These expressions satisfy the r equations from which they were obtained, and the remaining $q - r$ equations, identically in these $n - r$ variables. A numerical solution is obtained by assigning an arbitrary value to each of the remaining $n - r$ variables. All solutions of the q equations are obtained in this manner.*

It is to be noted that condition (58) of chapter 5 is the condition $r < n$, $r_a < n$, since $n = 3$. For the system (60) of chapter 5 it was proved that $r = 2$, $r_a = 2$. Hence the solution of these equations in chapter 5 illustrates theorem 6. For the system (67) of chapter 5 it was proved that $r = 1$, $r_a = 2$. The fact that this system has no solution illustrates theorem 6.

Other methods of proving the theorems in this chapter will be found in the references.

PROBLEMS

Discuss completely each of the following systems of equations

1.
$$\begin{aligned} x + 5y - z + 2u &= 1, \\ -10x + 9y - 10z + 5u &= 0, \\ -2x + 3y - 3u &= 2, \\ 6x + 7y + 3z &= 3, \\ 7x - y + 4z + u &= 0 \\ 3x + 2y - z + 5u &= -1 \end{aligned}$$

2.
$$\begin{aligned} x + 5y + z + 2u &= 1, \\ -2x + 7y + 6u &= 5, \\ -x + 3y - z + u &= 0, \\ 2x + y - 3u &= -1, \\ 3x + 7y + 3z + 3u &= 1 \end{aligned}$$

3.
$$\begin{aligned} -4x - y + 8z + 5u &= 2, \\ 9x - y + 5z + 16u &= -1, \\ -x + y + 4z &= 1, \\ 2x - y + 3z + 7u &= -1, \\ 5x + y - z + 2u &= 0 \end{aligned}$$

4.
$$\begin{aligned} -5x + 6y + 5z - 11u &= -1, \\ 2x + y - z &= 1, \\ -x + 3y + 4z - 2u &= -1, \\ 5x + y + 2z + 7u &= 0 \\ 2x + 3y + 7z + 5u &= -2 \end{aligned}$$

5.
$$\begin{aligned} x - y + 3z + u + 2v &= 12, \\ 2y - z + 3u + v &= 3, \\ -x - y + 2z + 2u + 6v &= 14, \\ x + 3y - 3z + u - 5v &= -11, \\ -2x + 5y + z + 2u - 4v &= -5, \\ -4x + y + 2z - 3u - 2v &= -9 \end{aligned}$$

6.
$$\begin{aligned} -x - 3y + 4z + 3u &= 0 \\ x - y + 2z + u &= 1, \\ x + y - z - u &= 4, \\ 3x - y + 3z + u &= 5 \\ 4x - 2y + 5z + 2u &= 2 \end{aligned}$$

7.
$$\begin{aligned} 7x + 2y - 2z + 3u &= 1, \\ x + 2y - z &= 5, \\ 3x + y + z + 4u &= 0, \\ 2x + y - 2z &= -1, \\ -x + 2y + u &= 1 \end{aligned}$$

8.
$$\begin{aligned} 4y + 7z - 2u + v &= 2, \\ x + y + 3z - 2u - v &= 0, \\ -7x + 3y - 2z - 3v &= -5, \\ -2x + z + 2v &= 1, \\ 5x + y + 2z + u + v &= 3, \\ -4x + 2y + z - u - v &= -2 \end{aligned}$$

9. $$\begin{aligned} 5x + y + u &= 3, \\ 5x + 8y + 5z - 2u &= -1, \\ 2x - y - z + u &= 1, \\ x + 3y + 2z - u &= 4, \\ 7x - z + 2u &= 2 \end{aligned}$$

10. $$\begin{aligned} 2x - y + z - u &= 1, \\ -2x + 5y + 2z + 3u + v &= 0, \\ x + y + 4z + 2u - v &= 3, \\ 5x - 2y + 3z + u + 2v &= -5, \\ -4x + 7y + 4z + 3u - 2v &= 2. \end{aligned}$$

11. $$\begin{aligned} x + y + 3z - u + 2v &= 5, \\ x - y + 2z + u &= 2, \\ -x - y - 2u + v &= 8, \\ 3x + y + 5z + 2u + v &= -1, \\ 5x + 3y + 11z + 5v &= 9, \\ -2y + 2z - u + v &= 10. \end{aligned}$$

12. $$\begin{aligned} x + 2y - 3z - u + v &= 7, \\ 3x + y + 5u - v &= 2, \\ 5x - z + 2u + v &= 4, \\ 3x + y - 4z - 4u + 3v &= 9, \\ 5y - 5z + u &= 12, \\ 7x - y - 2z - u + 3v &= 6. \end{aligned}$$

3. Linear homogeneous equations in n unknowns. If $k_1 = 0$, $\cdots$, $k_q = 0$ in (26), the equations are called *linear homogeneous equations in n unknowns.* Then the system of equations is

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0, \\ \cdot \qquad\quad & \quad \cdot \\ \cdot \qquad\quad & \quad \cdot \\ \cdot \qquad\quad & \quad \cdot \\ a_{q1}x_1 + \cdots + a_{qn}x_n &= 0. \end{aligned} \tag{85}$$

For this system it is true that $r_a = r$ because a minor which is in the a.m. but not in the c.m. has each element in its last column zero and therefore is zero. Now (85) is satisfied if each of $x_1, \cdots, x_n$ has the value zero. This solution is called *the zero solution* because it is the set of n zeros. It is also called the trivial solution.

It will be proved now that, if there is a solution of (85) which is not the zero solution, then $r < n$. This will be done by proving that, if $r = n$, then there is a contradiction. If $r = n$, then by theorems 6 and 2 there is exactly one solution, and it is the zero solution. This contradicts the hypothesis that there is a solution which is not the zero solution. It will now be proved, conversely,

that, if $r < n$, then there is a solution which is not the zero solution Since $r < n$, therefore $n - r \geq 1$ Also, in theorem 6 an arbitrary numerical value is assigned to each of $n - r$ variables Therefore here an arbitrary value is assigned to at least one variable Thus, for example, a solution is obtained if the value one is assigned to each of these $n - r$ variables This solution is not the zero solution Therefore the following important theorem has been proved

THEOREM 7 *A set of q linear homogeneous equations in n variables has a solution which is different from the zero solution if and only if the rank of the coefficient matrix of these equations is less than n A set of n linear homogeneous equations in n variables has a solution which is different from the zero solution if and only if the determinant of the coefficients in these equations is zero*

PROBLEMS

Discuss each of the following systems of equations

1 $$\begin{aligned} x + 2y + 5z - u &= 0 \\ 4x + 6y - 7z + 2u &= 0 \\ 2x - y - 4z + u &= 0 \\ 3x + 9y + 2z &= 0 \end{aligned}$$

2 $$\begin{aligned} x - y + 3z + u &= 0 \\ 11x - y - 2u &= 0 \\ 7x + 2y - 4z - u &= 0 \\ -2x + y + 2z + 3u &= 0 \end{aligned}$$

3 $$\begin{aligned} 2x - y + z - 3u &= 0 \\ 8x - 2y - 4z - 7u &= 0 \\ -x + y + z + 2u &= 0 \\ 5x - 4z - 2u &= 0 \end{aligned}$$

4 $$\begin{aligned} x - y + 2z + 3u &= 0 \\ 2x + 5y + z &= 0 \\ -x + 3y + 4z - 2u &= 0 \\ -3x + 2y - z + u &= 0 \\ 5x + y - z + 4u &= 0 \end{aligned}$$

5 $$\begin{aligned} 2x + y + 3z - u &= 0 \\ -x + z + 4u &= 0 \\ 3x + 5y - z + 2u &= 0 \\ x + 2y - 3u &= 0 \\ 4x - y + 2z + u &= 0 \end{aligned}$$

6. $$\begin{aligned} -x + 5y \phantom{{}+3z} + 9u &= 0,\\ 2x - y + 3z + 4u &= 0,\\ -5x + 4y + z - u &= 0,\\ x + y - 2z + 3u &= 0,\\ 5x - y - 8z + 3u &= 0. \end{aligned}$$

7. $$\begin{aligned} x - y + 2z \phantom{{}-2u} + v &= 0,\\ 3x + y + 5z - 2u &= 0,\\ -x + z + 2u - 7v &= 0,\\ 2x + 3y - 4z + u + 5v &= 0,\\ x - 3y + 12z - u - 11v &= 0,\\ 9x + 4y + 8z - 3u + 6v &= 0,\\ 5x - y + 13z - 7u + 2v &= 0. \end{aligned}$$

8. $$\begin{aligned} -5x + 2y - 10z + 5u &= 0,\\ 13x - y + 2z + 5u + 7v &= 0,\\ 2x + y - z + 3u + 4v &= 0,\\ -x + z + 4u - 3v &= 0,\\ 5x - y + 2z + 3u &= 0,\\ 3x + y + 7z + 2u + 5v &= 0. \end{aligned}$$

9. $$\begin{aligned} x + 3y - 7z + 2u + 4v &= 0,\\ x + 3y - 5z - u + 2v &= 0,\\ 3x + 2y + u - v &= 0,\\ 2x - y + 3z + 5u - v &= 0,\\ -x + 4y - 8z - 6u + 3v &= 0,\\ 4x + 5y - 3z - 3u - v &= 0. \end{aligned}$$

10. $$\begin{aligned} 3x + 7y + 3z + v &= 0,\\ 4x + 7y + 10z + u - 10v &= 0,\\ 3y - 2z + u + 4v &= 0,\\ x - y + z - 2u &= 0,\\ 2x + 5y + 4z + u - 3v &= 0,\\ -3x - y - 7z + 2u + 7v &= 0,\\ 3x + 14y + 5z + 5u - 2v &= 0. \end{aligned}$$

In the proof of theorem 6 one method of obtaining all solutions of equations (85) is exhibited, because each set of arbitrary values of the $n - r$ transposed variables determines a solution, and each solution can be obtained in this way. Another method of obtaining all solutions will now be illustrated by means of the following set of five numerical linear homogeneous equations in four unknowns:

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 0,\\ 2x_1 - x_2 - 2x_3 + x_4 &= 0,\\ -x_1 - 4x_2 + 7x_3 - 5x_4 &= 0,\\ 8x_1 - 7x_2 - 4x_3 + x_4 &= 0,\\ 5x_1 - 4x_2 - 3x_3 + x_4 &= 0. \end{aligned} \tag{86}$$

By lemma 1 and lemma 2 it is proved that $r = 2$, $r_a = 2$ By theorem 6 it is sufficient to solve the first two of equations (86) Thus, for example, the solution for x_1 and x_2 is

$$(87) \qquad x_1 = \tfrac{5}{3}x_3 - x_4 \quad x_2 = \tfrac{4}{3}x_3 - x_4$$

If the arbitrary values 0 and 1 are assigned to x_3 and x_4 respectively, then the solution -1, -1, 0 1 is obtained If the arbitrary values 3 and 0 are assigned to x_3 and x_4 respectively, then the solution 5 4 3 0 is obtained

A method of obtaining all solutions from these two particular solutions will be explained next The first step in the explanation of this new method is to show that, if c_1, c_2, c_3, c_4 is a solution of (86), and if m is an arbitrary number, then mc_1, mc_2, mc_3, mc_4 is also a solution of (86) Thus since c_1, c_2, c_3, c_4 satisfy (86_2), it is true that $2c_1 - c_2 - 2c_3 + c_4 = 0$ Hence $m(2c_1 - c_2 - 2c_3 + c_4) = 0$ Hence $2(mc_1) - (mc_2) - 2(mc_3) + (mc_4) = 0$ Hence mc_1 mc_2 mc_3 mc_4 satisfy (86_2) Similarly it is proved that they satisfy each of equations (86) The second step in this new method is the proof that if c_1 c_2 c_3 c_4 and d_1, d_2 d_3, d_4 are two solutions of (86), then $c_1 + d_1$ $c_2 + d_2$ $c_3 + d_3$, $c_4 + d_4$ is a solution of (86) Thus, since c_1 c_2 c_3 c_4 and d_1, d_2, d_3, d_4 satisfy (86_1), it is true that $c_1 - 2c_2 + c_3 - c_4 = 0$ and that $d_1 - 2d_2 + d_3 - d_4 = 0$ Hence $(c_1 + d_1) - 2(c_2 + d_2) + (c_3 + d_3) - (c_4 + d_4) = 0$ Hence $c_1 + d_1, c_2 + d_2$ $c_3 + d_3$ $c_4 + d_4$ satisfy (86_1) Similarly it is proved that they satisfy each of equations (86)

Finally let m_1 and m_2 be arbitrary numbers Since -1, -1, 0, 1 is a solution of (86) therefore $m_1(-1)$, $m_1(-1)$, m_1 0, m_1 1 is a solution Also, since 5 4 3 0 is a solution, therefore m_2 5, m_2 4 m_2 3 m_2 0 is a solution Hence also $m_1(-1) + m_2$ 5, $m_1(-1) + m_2$ 4 m_1 0 + m_2 3, m_1 1 + m_2 0 is a solution This is a very important property of the particular solutions -1, -1, 0, 1 and 5, 4 3, 0 Another very important property of these particular solutions is that every solution can be obtained in this manner from them This will be proved in theorem 8 It will also be proved that neither of these two particular solutions can be so obtained from the other Finally, it will also be proved that many pairs of particular solutions have these same properties, which the pair -1 -1 0, 1 and 5, 4, 3, 0 of solutions have, with regard to equations (86)

To illustrate the discussion which will be given later if n is arbitrary the preceding discussion of the numerical equations (86) will be summarized in terms of the new ideas of vectors and linear dependence. An ordered set of n numbers is called an *n-vector.* In the preceding discussion n is four, and each solution of equations (86) is an n-vector. If the solution (c_1, c_2, c_3, c_4) is designated by γ, then the solution (mc_1, mc_2, mc_3, mc_4) is designated by $m\gamma$. The vector $m\gamma$ is called a *scalar multiple of the vector* γ. If the solution (d_1, d_2, d_3, d_4) is designated by δ, then the solution $(c_1 + d_1, c_2 + d_2, c_3 + d_3, c_4 + d_4)$ is designated by $\gamma + \delta$. It is not true in general that $(c_1d_1, c_2d_2, c_3d_3, c_4d_4)$ is also a solution. Thus, for example, the vector so formed from the particular solutions $(-1, -1, 0, 1)$ and $(5, 4, 3, 0)$ is $(-5, -4, 0, 0)$ but $-5, -4, 0, 0$ do not satisfy (86_1). Therefore, as solutions of linear equations vectors are added, but they are not multiplied. However, as noted earlier, there is a scalar multiplication of vectors by numbers.

The set of all solutions of (86) is an instance of a *linear space of n dimensions* because it has these two properties that the sum of two members of the set is also a member of the set and that the product of a number and a member of the set is also a member of the set. Such a set is also called a *vector space of n dimensions.* Another property which a linear space of n dimensions has, by definition, is that the n-vector each of whose elements is zero is a member of the space. This vector is often designated merely by 0 and is called the *zero vector.* The zero vector is a solution of every system of homogeneous equations in n variables. If at least one of $k_1, k_2, \cdots, k_q$ is not zero, then the zero vector is not a solution of the system. Therefore the following discussion applies to the solution of homogeneous equations, but it does not apply to the solution of non-homogeneous equations.

It was proved that, if γ and δ are two solutions of (86), and if m_1 and m_2 are two numbers, then $m_1\gamma + m_2\delta$ is a solution of (86). The vector $m_1\gamma + m_2\delta$ is called *a linear combination of the vectors γ and δ.* It will be proved later that, if ζ is a solution of (86), then there are numbers m_3 and m_4 such that $\zeta = m_3(-1, -1, 0, 1) + m_4(5, 4, 3, 0)$; that is, ζ is a linear combination of $(-1, -1, 0, 1)$ and $(5, 4, 3, 0)$. This is one of the reasons why $(-1, -1, 0, 1)$ and $(5, 4, 3, 0)$ are said to form *a fundamental set of solutions* of (86). The other reason is that neither $(5, 4, 3, 0)$ nor $(-1, -1, 0, 1)$ is

a multiple of the other This last fact can also be stated by saying that, if m_1 and m_2 are two numbers such that $m_1(-1, -1\ 0\ 1) + m_2(5, 4\ 3\ 0)$ is the zero vector $(0\ 0\ 0\ 0)$ then $m_1 = 0$ and $m_2 = 0$ In general two vectors γ and δ are said to be *linearly independent* precisely when from the fact that $m_1\gamma + m_2\delta$ is the zero vector it follows that $m_1 = 0$ and $m_2 = 0$ This means that two vectors γ and δ are *linearly dependent* precisely when there are two numbers m and m at least one of which is not zero such that $m\gamma + m\,\delta$ is the zero vector Similar definitions are made for more than two vectors A fundamental set of solutions of (86) is also called a *basis of the set of solutions* of (86)

PROBLEMS

Solve each of the systems having $r < n$ in the problems on page 212 for r of the variables in terms of the remaining $n - r$ variables Obtain one particular solution by assigning the value 1 to one of the transposed variables and 0 to the other transposed variables Obtain a second particular solution by assigning the value 1 to a different one of the transposed variables and 0 to the other transposed variables Repeat the process until the value 1 has been assigned to each of the transposed variables in turn Tabulate these $n - r$ particular solutions

The following lemmas will be used in the proof of the fundamental theorem 8 which states that if r is less than n for the equations (85) then there is a fundamental set of solutions of (85) and that this fundamental set consists of $n - r$ linearly independent solutions It is to be noted especially that these lemmas are general statements concerning ordered sets with n numbers in a set and that it is not assumed in the proof of these lemmas that the vectors are solutions of (85)

If the n-vector $c_1\ c_2 \quad c_n$ is designated by ζ_i then p n-vectors $\zeta_1 \quad , \zeta_p$ determine the matrix

$$(88) \qquad \begin{bmatrix} c_{11} & c_{12} & & c_{1n} \\ c_2 & c_{22} & & c_{2n} \\ & & & \\ & & & \\ c_{p1} & c_{p2} & & c_{pn} \end{bmatrix}$$

LEMMA 3 *If p and n are positive integers such that $p \leq n$ and if p n vectors are linearly dependent then the rank of the matrix of these vectors is less than p*

PROOF. The linearly dependent vectors can be rearranged so that, after the notation $\zeta_1, \cdots, \zeta_p$ is assigned, there are constants $m_1, \cdots, m_p$ such that $m_1 \neq 0$ and $m_1\zeta_1 + \cdots + m_p\zeta_p$ is the zero vector. Therefore

$$\begin{aligned} m_1c_{11} + m_2c_{21} + \cdots + m_pc_{p1} &= 0, \\ m_1c_{12} + m_2c_{22} + \cdots + m_pc_{p2} &= 0, \\ &\cdots \\ m_1c_{1n} + m_2c_{2n} + \cdots + m_pc_{pn} &= 0. \end{aligned} \tag{89}$$

Let D designate the particular p-rowed minor of (88) whose symbol is

$$\begin{vmatrix} c_{11} & \cdots & c_{1p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ c_{p1} & \cdots & c_{pp} \end{vmatrix}. \tag{90}$$

Let D_1 designate the determinant whose symbol is

$$\begin{vmatrix} m_1c_{11} & \cdots & m_1c_{1p} \\ c_{21} & \cdots & c_{2p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ c_{p1} & \cdots & c_{pp} \end{vmatrix}. \tag{91}$$

Therefore $D_1 = m_1D$. Also $m_1 \neq 0$ If it is proved that $D_1 = 0$, it will follow that $D = 0$. Let D_2 designate the determinant

$$\begin{vmatrix} m_1c_{11} + m_2c_{21} & \cdots & m_1c_{1p} + m_2c_{2p} \\ c_{21} & \cdots & c_{2p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ c_{p1} & \cdots & c_{pp} \end{vmatrix}. \tag{92}$$

Then $D_2 = D_1$. Again, if D_3 is obtained from (92) by adding to the first row of (92) m_3 times the third row of (92), then $D_3 = D_2$. If this process is continued, there is obtained a determinant whose

symbol has in the first row and the jth column the number $m_1c_{1j} + \cdots + m_pc_{pj}$. Hence by (89) the first row is a row of zeros. Therefore this determinant is zero and $D = 0$.

In the same way it is proved that each p-rowed minor of (88) is zero.

PROBLEMS

1. If $\zeta_1 = (1\ 2\ 5\ 1)$ and $\zeta_2 = (3\ -2\ 1\ -7)$ find ζ_3 such that $3\zeta_1 + \zeta_2 - 2\zeta_3 = 0$. Find the rank of the matrix formed by these three vectors and thus illustrate lemma 3.

2. Proceed as in problem 1 if ζ_1, ζ_2 and ζ_4 are such that $-2\zeta_1 + \zeta_2 - \zeta_4 = 0$.

3. Proceed as in problem 1 if $\zeta_1 = (-1\ 3\ -2\ 1)$, $\zeta_2 = (1\ 1\ 6\ -5)$ and ζ_3 is such that $3\zeta_1 + 2\zeta_2 - \zeta_3 = 0$.

4. Proceed as in problem 3 if ζ_1, ζ_3 and ζ_4 are such that $\zeta_1 + \zeta_2 - 4\zeta_4 = 0$.

5. If $\zeta_1 = (1\ 0\ -3\ 2\ -1)$, $\zeta_2 = (2\ 3\ -1\ 0\ -1)$, $\zeta_3 = (0\ -1\ 3\ -2\ 1)$ find ζ_4 such that $2\zeta_1 + \zeta_2 + 5\zeta_3 - 2\zeta_4 = 0$. Find the rank of the matrix formed by these four vectors and thus illustrate lemma 3.

6. Proceed as in problem 5 if $\zeta_1 = (-1\ 1\ 2\ 5\ 1)$, $\zeta_2 = (0\ 2\ -1\ 1\ -1)$, $\zeta_3 = (0\ -4\ 1\ -9\ -1)$ and ζ_4 is such that $2\zeta_1 + 3\zeta_2 + \zeta_3 - 2\zeta_4 = 0$.

7. If $\zeta_1 = (1\ \ 1\ 1\ 1\ 0)$, $\zeta_2 = (2\ -1\ 4\ 3\ -2)$, $\zeta_3 = (1\ 4\ 1\ -2\ -2)$ find ζ_4 such that $\zeta_1 + \zeta_2 + \zeta_3 - 2\zeta_4 = 0$. Find the rank of the matrix of these four vectors and thus illustrate lemma 3.

8. Proceed as in problem 7 for ζ_1, ζ_2, ζ_3 and ζ_5 such that $3\zeta_1 - \zeta_2 - \zeta_3 + 2\zeta_5 = 0$.

9. Proceed as in problem 7 if $\zeta_1 = (1\ 0\ -1\ 2\ -5)$, $\zeta_2 = (-1\ 3\ -1\ 1\ 0)$, $\zeta_5 = (2\ -1\ 0\ 1\ 1)$ and ζ_3 is such that $\zeta_1 + \zeta_2 - 2\zeta_3 + \zeta_5 = 0$.

10. Proceed as in problem 9 for ζ_1, ζ_2, ζ_4, ζ_5 such that $3\zeta_1 - \zeta_2 - 2\zeta_4 + 3\zeta_5 = 0$.

LEMMA 4. *If p and n are positive integers such that $p \leq n$ and if the rank of a matrix of p rows and n columns is less than p, then the p n-vectors which constitute the rows of this matrix are linearly dependent.*

PROOF. Let r be the rank of the matrix. The notation (88) is assigned after the rows and columns of the given matrix have been rearranged so that in (88) there is a non zero r-rowed minor in the upper left-hand corner. Let the n-vector which is the first row of (88) be designated by ζ_1. In general let ζ_i designate the n-vector which is the ith row of (88).

It will now be proved that $\zeta_1, \ldots, \zeta_p$ are linearly dependent. This will be done by exhibiting numbers $m_1, \ldots, m_p$ such that $m_p \neq 0$ and that $m_1\zeta_1 + \cdots + m_p\zeta_p = 0$. Thus each of equations

(89) will be proved. The effective numbers $m_1, \cdots, m_p$ will be found by consideration of an auxiliary determinant B whose symbol is

$$(93) \qquad \begin{vmatrix} c_{11} & \cdots & c_{1r} & c_{1n} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ c_{r1} & \cdots & c_{rr} & c_{rn} \\ c_{p1} & \cdots & c_{pr} & c_{pn} \end{vmatrix}.$$

Let $M_1, \cdots, M_r, M_p$ designate the minors of the elements of the last column of B. It is to be noted especially that the notation was chosen originally so that $M_p \neq 0$. Expansion of B by its last column gives

$$(94) \qquad B = \sum_{i=1}^{r} (-1)^{i+r+1} c_{in} M_i + (-1)^{r+1+r+1} c_{pn} M_p.$$

Also, $B = 0$, since B is an $(r + 1)$-rowed minor of a matrix of rank r. Let $m_1, \cdots, m_p$ be defined by

$$(95) \qquad \begin{aligned} m_i &= (-1)^{i+r+1} M_i \quad && (i = 1, \cdots, r), \\ m_i &= 0 && (r < i < p), \\ m_p &= M_p. \end{aligned}$$

Therefore (94) becomes

$$(96) \qquad 0 = m_1 c_{1n} + \cdots + m_p c_{pn}.$$

Therefore the last equation in (89) has been proved.

Similarly, if j is an integer such that $1 \leqq j \leqq n$ and if B_j is defined by replacing the last column of B by $c_{1j}, \cdots, c_{rj}, c_{pj}$, then $B_j = 0$, either because it is an $(r + 1)$-rowed minor of (88) or because it has two columns alike. Also, for each value of j the minors of the elements of the last column of B_j are $M_1, \cdots, M_r, M_p$. Hence, *for the same values of* $m_1, \cdots, m_p$ it is true that

$$(97) \qquad 0 = m_1 c_{1j} + \cdots + m_p c_{pj} \quad (j = 1, \cdots, n).$$

Since equations (97) are precisely equations (89), it has been proved that $\zeta_1, \cdots, \zeta_p$ are linearly dependent. It is to be noted that it has been proved, in fact, that $\zeta_1, \cdots, \zeta_r, \zeta_p$ are linearly dependent. This is true because (93) involves only $\zeta_1, \cdots, \zeta_r, \zeta_p$.

It is evidenced also by $m_s = 0$ $(r < s < p)$ in (95) Similarly it is proved that if $r < s < p$ then $\zeta_1 \quad \zeta_r \ \zeta_s$ are linearly dependent The coefficients $m_1 \quad m_r \ m_s$ in this dependence are obtained by using a determinant formed by replacing the last row of (93) by $c_{s1} \quad c_{sr} \ c_{sn}$

PROBLEMS

In each of the following problems show that the rank r of the matrix formed by the p vectors is less than p Find r of the vectors on which each of the remaining $p - r$ vectors is linearly dependent Exhibit each such dependence as an equation

1 $\zeta_1 = (1\ -1\ 2\ 1)$ $\zeta_2 = (0\ -1\ 1\ 3)$ $\zeta_3 = (1\ 3\ 0\ -1)$
$\zeta_4 = (2\ -5\ 5\ 1)$

2 $\zeta_1 \quad (2 \quad 1\ 1\ 1)$ $\zeta_2 = (-1\ 5\ 2\ 1)$ $\zeta_3 = (0\ 1\ 1\ 1)$ $\zeta_4 = (0\ -2\ 0\ 1)$

3 $\zeta_1 = (-1\ 1\ 0\ 2)$ $\zeta_2 = (1\ 3\ 2\ -4)$ $\zeta_3 = (0\ 2\ 1\ -1)$
$\zeta_4 = (2\ 0\ 1\ -2)$

4 $\zeta_1 \quad (\quad 1\ 1\ 1\ 1)$ $\zeta_2 = (1\ -3\ 0\ 2)$ $\zeta_3 = (2\ -4\ -1\ 1)$
$\zeta_4 = (1\ 1\ -2 \quad 4)$

5 $\zeta_1 = \ 3\ 1\ 2\ -1\ 0)$ $\zeta_2 = (-1\ 1\ 0\ 3\ 2)$ $\zeta_3 = (5\ 3\ 1\ -1\ -1)$
$\zeta_4 = (2\ 1\ 0\ -1\ -1)$ $\zeta_5 = (-5\ 1\ -5\ 5\ 1)$

6 $\zeta_1 = (5\ -1\ 2\ 0\ -2)$ $\zeta_2 = (1\ 0\ -2\ -1\ 3)$ $\zeta_3 = (2\ 1\ 4\ -3\ 1)$
$\zeta_4 = (1\ 2\ 4\ -5\ 4)$ $\zeta_5 = (3\ 0\ 4\ -1 \quad 2)$

7 $\zeta_1 = (5\ 2\ 2\ -8\ -4\ -1\ -1)$ $\zeta_2 = (1\ -1\ 0\ 2\ 1\ 3\ -1)$
$\zeta_3 = (3\ 1\ 1 \quad 1\ 0\ 2\ 1)$ $\zeta_4 = (2\ 1\ 2\ -5 \quad 1\ -1\ 0)$
$\zeta_5 = (\quad 1\ 1\ 1\ 0\ 2\ -1\ 3)$ $\zeta_6 = (-3\ 2\ 1\ -8\ -3\ -9\ 1)$

8 $\zeta_1 = (-1 \quad 5\ 1\ 3\ -2\ 1\ 1)$ $\zeta_2 = (2\ 1\ -1\ 0\ 2\ 1\ -1)$
$\zeta_3 = (\quad 1\ 3\ 2\ 1\ 0\ -1\ 0)$ $\zeta_4 = (-1\ -2\ 2\ 3\ 1\ 5\ -1)$
$\zeta_5 = (1\ 0\ -1\ -1\ 3\ 5\ -2)$ $\zeta_6 = (-1\ 2\ 4\ 5\ 0\ 0\ 0)$

LEMMA 5 *Let p and n be positive integers such that $p \leq n$ Then p n vectors are linearly independent if and only if the rank of the matrix of these vectors is p*

PROOF The statement that p n vectors are linearly independent only if the rank of the matrix of these vectors is p means that if the p vectors are linearly independent then the rank is p This statement will now be proved This will be done by showing that if the vectors are linearly independent and if the rank is less than p then there is a contradiction By lemma 4, if the rank is less than p the vectors are dependent and there is a contradiction

Next it will be proved that if the rank is p then the vectors are linearly independent This will be done by showing that if the rank is p and if the rows of the matrix are linearly dependent then there is a contradiction By lemma 3 if the rows are de-

pendent, then the rank is less than p, and there is a contradiction. This completes the proof of lemma 5.

It is to be noted especially that $p \leqq n$ in lemmas 3, 4, and 5. Lemma 6, in which $p > n$, will now be proved. From the matrix (88) form a new matrix A by adjoining $p - n$ columns, each adjoined column consisting entirely of zeros. Let α_1 designate the p-vector which is the first row of the matrix A. Then $\alpha_1 = (c_{11}, c_{12}, \cdots, c_{1n}, 0, \cdots, 0)$, in which the last $p - n$ elements are zeros. In general, define $\alpha_i = (c_{i1}, c_{i2}, \cdots, c_{in}, 0, \cdots, 0)$, if $i = 1, \cdots, p$. Then $\alpha_1, \cdots, \alpha_p$ is a set of p p-vectors. Now the rank of A is less than p, since each p-rowed minor of A has at least one column of zeros. Therefore, by lemma 4, with p and n replaced by p and p respectively, it is true that $\alpha_1, \cdots, \alpha_p$ are linearly dependent. This means, by definition, that there are numbers $m_1, \cdots, m_p$, at least one of which is not zero, such that $m_1\alpha_1 + \cdots + m_p\alpha_p$ is the zero p-vector. This means that equations (89) hold, and also $p - n$ equations formed similarly from the last $p - n$ columns of A hold. Each of these last $p - n$ equations is the equation $m_1 \cdot 0 + \cdots + m_p \cdot 0 = 0$, since each of the last $p - n$ columns consists entirely of zeros. These last $p - n$ equations therefore give no information. However, the fact that equations (89) hold means precisely that $m_1\zeta_1 + \cdots + m_p\zeta_p = 0$. This states that $\zeta_1, \cdots, \zeta_p$ are linearly dependent. Thus the proof of lemma 6 is completed.

LEMMA 6. *If p and n are positive integers such that $p > n$, then p n-vectors are linearly dependent.*

These lemmas will now be used to prove that, if the rank r of the q linear homogeneous equations (85) is less than n, then there is a set of $n - r$ linearly independent solutions of (85). As in the proof of theorem 6, $n - r$ of the variables are transposed, and r of the equations are solved for the remaining variables in terms of the transposed variables. The equations and variables can be rearranged so that in the notation (85) the r equations to be solved are the first r equations and that the transposed variables are $x_{r+1}, \cdots, x_n$. Now let the numbers $d_{1,r+1}, \cdots, d_{1n}$ be assigned to $x_{r+1}, \cdots, x_n$ respectively, and let the values of $x_1, \cdots, x_r$ be computed. Let these values be designated by $d_{11}, \cdots, d_{1r}$ respectively. Then $d_{11}, \cdots, d_{1r}, d_{1,r+1}, \cdots, d_{1n}$ is a solution of (85).

Now it will be proved that the numbers $d_{j\,r+1}, \quad , d_{jn}$ ($j = 1, \quad , n - r$) may be assigned so that the determinant D whose symbol is

$$(98) \qquad \begin{vmatrix} d_{1\,r+1} & & d_{1n} \\ & & \\ d_{n-r\,r+1} & & d_{n-r\,n} \end{vmatrix}$$

is not zero One way of assigning these values so that $D \neq 0$ is that in which the diagonal elements in (98) are ones and the non diagonal elements are zeros Then for these values it is true that $D \neq 0$ Now consider the matrix

$$(99) \qquad \begin{bmatrix} d_{11} & & d_{1r} & d_{1\,r+1} & & d_{1n} \\ & & & & & \\ d_{n-r\,1} & & d_{n-r\,r} & d_{n-r\,r+1} & & d_{n-r\,n} \end{bmatrix}$$

of these $n - r$ solutions By lemma 5 with p, n there replaced here by $n - r$ n respectively it is true that *these $n - r$ solutions are linearly independent* These solutions will be designated by $\delta_1 \quad , \delta_{n-r}$ respectively Thus δ_i is the n-vector which is the ith row of the matrix (99)

PROBLEMS

For each of the following systems of equations find the rank r of the coefficient matrix and solve the equations If $r < n$ find a set of $n - r$ linearly independent solutions of the system

1 $x + 2y - 3z \quad u = 0$
$x - y + 2z - 2u = 0$
$x + 5y - 8z \quad = 0$
$2x - 5y + 9z - 5u = 0$

2 $x + 3y - 2z + 4u = 0$
$2x + y + z + u = 0$
$3x - y + 4z - 2u = 0$
$7x + 6y + z + 7u = 0$

3 $2x + 3y + z - u = 0$
$x + y - z + 5u = 0$
$3x + 5y + 3z - 7u = 0$
$x + 2y + 2z - 6u = 0$
$5x + 6y - 2z + 14u = 0$

4. $$\begin{aligned} 5x + y + z + u &= 0,\\ 2x - y - 3z + 2u &= 0,\\ -x + 4y + 10z - 5u &= 0,\\ x + 3y + 7z - 3u &= 0,\\ 7x - 2z + 3u &= 0. \end{aligned}$$

5. $$\begin{aligned} x - 2y + z - 5u &= 0,\\ -x + 3y - 2z + u &= 0,\\ 3x - 6y + 6z - 10u &= 0,\\ x + y + z - 12u &= 0,\\ 2y + z - 3u &= 0. \end{aligned}$$

6. $$\begin{aligned} 6x - 3y - 2z - 3u &= 0,\\ 3x + 6y - z + 9u &= 0,\\ 3x + y - z + 2u &= 0,\\ -x + 3y + 2z + 5u &= 0,\\ 2x - y + z &= 0. \end{aligned}$$

7. $$\begin{aligned} x - y + z + 3u - v &= 0,\\ -2x + y + z - u + 3v &= 0,\\ 5x + y + 2z - v &= 0,\\ -5x - y + 3z + 3u + 5v &= 0,\\ x + 3z + u + 2v &= 0,\\ -4x + y + 3v &= 0. \end{aligned}$$

8. $$\begin{aligned} 2x - y + z + u + 3v - w &= 0,\\ 4x - 3u + 2v - 4w &= 0,\\ x + 3y - z - u + 2w &= 0,\\ 2x - 3y + 3z + 6u + 7v + w &= 0,\\ -x + 2y + 3u + v + 5w &= 0,\\ x - 9y + 3z + 2v - 10w &= 0. \end{aligned}$$

9. $$\begin{aligned} x - y - 2z + 3u + v - w &= 0,\\ -4x - 2y - 6z + 4u + v - 3w &= 0,\\ 2x + y + 5z - v + w &= 0,\\ 8x + y + 7z + 2u + 2w &= 0,\\ 3x - 3y - 10z + 5u + 4v - 2w &= 0,\\ -3x + z + u - v - w &= 0. \end{aligned}$$

10. $$\begin{aligned} 2x + y - z - 3u + 5v &= 0,\\ x - 6y + 3z + 5u - v &= 0,\\ 3x - 4y + z + 2u - v &= 0,\\ -x + 2y - u + 3v &= 0,\\ x - 3y + z + u + 2v &= 0,\\ 3x + y - 2z - 6u + 10v &= 0. \end{aligned}$$

It will now be proved that each solution of (85) is a linear combination of these particular $n - r$ solutions, $\delta_1, \cdots, \delta_{n-r}$. This will be done by proving, more generally, that, if $\delta_1, \cdots, \delta_{n-r}$ is any set of $n - r$ linearly independent solutions of (85), and, if δ

is any solution of (85), then δ is linearly dependent on $\delta_1, \cdots, \delta_{n-r}$. Since $\delta_1, \cdots, \delta_{n-r}$ is a set of $n - r$ linearly independent solutions, by lemma 5 there is an $(n - r)$-rowed, non-zero minor in their matrix. Let the notation

$$\begin{aligned} \delta &= (d_1, \cdots, d_r, d_{r+1}, \cdots, d_n), \\ \delta_i &= (d_{i1}, \cdots, d_{ir}, d_{i\,r+1}, \cdots, d_{in}) \quad (i = 1, \cdots, n - r) \end{aligned} \tag{100}$$

be chosen so that (98) is this non-zero minor. Let $\alpha, \alpha_1, \cdots, \alpha_{n-r}$ be defined by

$$\begin{aligned} \alpha &= (d_{r+1}, \cdots, d_n) \\ \alpha_i &= (d_{i\,r+1}, \cdots, d_{in}) \quad (i = 1, \cdots, n - r) \end{aligned} \tag{101}$$

Since (98) is not zero, it is true by lemma 5 with p replaced by $n - r$ and n replaced by $n - r$, that $\alpha_1, \cdots, \alpha_{n-r}$ are linearly independent. Also by lemma 6 with n replaced by $n - r$ and p by $n - r + 1$, it is true that $\alpha, \alpha_1, \cdots, \alpha_{n-r}$ are linearly dependent and, in fact, that α is linearly dependent on $\alpha_1, \cdots, \alpha_{n-r}$. Hence there are numbers $m_1, \cdots, m_{n-r}$ such that

$$\begin{aligned} d_{r+1} &= m_1 d_{1\,r+1} + \cdots + m_{n-r} d_{n-r\,r+1}, \\ &\cdots\cdots \\ d_n &= m_1 d_{1n} + \cdots + m_{n-r} d_{n-r\,n} \end{aligned} \tag{102}$$

These equations may be summarized by

$$d_j = m_1 d_{1j} + \cdots + m_{n-r} d_{n-r\,j} \quad (j = r + 1, \cdots, n) \tag{103}$$

Next it will be proved that if $j = 1, \cdots, r$, then equations similar to (103) hold and that the *same numbers* $m_1, \cdots, m_{n-r}$ *appear in the new equations*. Now in the proof of theorem 6 it was found that x_1 is a linear homogeneous function of the transposed variables $x_{r+1}, \cdots, x_n$ if the original equations are homogeneous. Let the notation $x_1 = b_{1\,r+1}x_{r+1} + \cdots + b_{1n}x_n$ be used. In general, there are constants b_{st} such that

$$x_s = b_{s\,r+1}x_{r+1} + \cdots + b_{sn}x_n \quad (s = 1, \cdots, r) \tag{104}$$

Since $d_1, \cdots, d_r, d_{r+1}, \cdots, d_n$ is by hypothesis a solution of (85), it is true that

$$d_s = b_{s\,r+1}d_{r+1} + \cdots + b_{sn}d_n \quad (s = 1, \cdots, r) \tag{105}$$

Also, since $(d_{11}, \cdots, d_{1r}, d_{1,r+1}, \cdots, d_{1n})$ is a solution,

$$(106) \quad d_{1s} = b_{s,r+1}d_{1,r+1} + \cdots + b_{sn}d_{1n} \quad (s = 1, \cdots, r).$$

In general, since $(d_{i1}, \cdots, d_{ir}, d_{i,r+1}, \cdots, d_{in})$ is, for each value of i from 1 to $n - r$, a solution of (85), it is true that

$$(107) \quad d_{is} = b_{s,r+1}d_{i,r+1} + \cdots + b_{sn}d_{in}$$

$$(s = 1, \cdots, r; i = 1, \cdots, n - r).$$

Now equations (103) are to be multiplied by b_{1j} respectively, and the results added. The left-hand side of the resulting last equation will be $d_{r+1}b_{1,r+1} + \cdots + d_nb_{1n}$. By (105) with $s = 1$ this number is d_1. The coefficient of m_1 on the right-hand side of that equation will be $d_{1,r+1}b_{1,r+1} + \cdots + d_{1n}b_{1n}$. By (106) with $s = 1$ this number is d_{11}. In general, the coefficient of m_i on the right-hand side will be $d_{i,r+1}b_{1,r+1} + \cdots + d_{in}b_{1n}$. By (107) with $s = 1$ this number is d_{i1}. Hence it has been proved that

$$(108) \qquad d_1 = m_1d_{11} + \cdots + m_{n-r}d_{n-r,1}.$$

Similarly it is proved, by multiplying equations (103) by b_{2j} respectively and adding the results, that

$$d_2 = m_1d_{12} + \cdots + m_{n-r}d_{n-r,2}.$$

And in general it is proved similarly that

$$(109) \quad d_j = m_1d_{1j} + \cdots + m_{n-r}d_{n-r,j} \quad (j = 1, \cdots, r).$$

By the notations (100) for δ and $\delta_1, \cdots, \delta_{n-r}$ equations (103) and (109) show that the vector δ is a linear combination of $\delta_1, \cdots, \delta_{n-r}$. This completes the proof of the following fundamental theorem.

A fundamental set of solutions is a set which consists of linearly independent solutions and which has the property that an arbitrary solution is a linear combination of these linearly independent solutions.

THEOREM 8. *If the rank r of q linear homogeneous equations in n variables is less than n, then there is a fundamental set of solutions which consists of $n - r$ solutions. This fundamental set consists of the n-vectors $\delta_1, \cdots, \delta_{n-r}$, in which δ_i is the solution obtained by*

assigning the value 1 to the ith transposed variable and the value zero to each of the other transposed variables

It can easily be proved that there are many fundamental sets of solutions and that each fundamental set consists of $n - r$ solutions

PROBLEMS

1 For each problem on page 222 find a fundamental set of solutions which is different from the fundamental set found there Express each solution in this second fundamental set as a linear combination of the solutions in the first fundamental set

2 In problem 1 express each solution in the first fundamental set as a linear combination of the solutions in the second fundamental set

For each of the following systems find the rank r of the coefficient matrix and solve the equations If $r < n$, find a fundamental set of solutions Then find a different fundamental set Express each solution in the first fundamental set as a linear combination of the solutions in the second fundamental set

3
$$\begin{aligned} x - 2y + z - u + v &= 0, \\ x + 3y - 5z + 2u &= 0, \\ 5x + 5y - 13z + 4u + 2v &= 0, \\ 3x + 4y - 9z + 3u + v &= 0 \\ 5y - 6z + 3u - v &= 0 \end{aligned}$$

4
$$\begin{aligned} 2x + 2y + 4z - 3u &= 0, \\ x + 2y - z + 3u &= 0, \\ 2x - y + u &= 0, \\ 8x - y + 7u &= 0, \\ 3x - y + z + 2u &= 0, \\ 2x + 4y + 2z - u &= 0 \end{aligned}$$

5
$$\begin{aligned} 2x + y + z + 3u &= 0, \\ 3x + y - z + 4u &= 0, \\ x - y + 2u &= 0, \\ -2y + 3z + 8u &= 0, \\ 3x - 2z + u &= 0, \\ -x - y + 2u &= 0 \end{aligned}$$

6
$$\begin{aligned} 2x + y + 7z - u + 3v &= 0, \\ x - y - 2z + u + 5v &= 0, \\ 5x - 2y + z + 2u + 18v &= 0, \\ 7x + 5y + 30z - 5u + 7v &= 0, \\ 3x + 3y + 16z - 3u + v &= 0 \end{aligned}$$

7
$$\begin{aligned} x - 2y + z + 3u - v &= 0, \\ 2x + 3y - z + u &= 0, \\ -x + 5y + 4u + 7v &= 0, \\ 6x - y - z + u - 8v &= 0, \\ x - 12y + 3z + u - 9v &= 0, \\ 5x - 3y + 2z + 10u - 3v &= 0 \end{aligned}$$

8
$$\begin{aligned} 3x - y + 2z + u + 5v &= 0, \\ y - 3z - u + 2v &= 0, \\ x - 2y + 5z - u &= 0, \\ x + 2y - 5z + 4u + 3v &= 0, \\ 5x + y - 4z + 2u + 12v &= 0, \\ 11x - 7y + 16z + u + 15v &= 0. \end{aligned}$$

Other proofs of these theorems and other facts about homogeneous equations are in the references cited at the end of this book

4 Geometrical interpretation if the number of variables is two or three Equations (26) may be written more simply if $n = 2$ Thus, if x_1 is replaced by x and x_2 by y, and if a_{i1} is replaced by

a_{i1} and a_{i2} by b_i, then these equations become

$$
\begin{aligned}
a_1x + b_1y &= k_1,\\
a_2x + b_2y &= k_2,\\
\cdot \quad \cdot \quad &\cdot\\
\cdot \quad \cdot \quad &\cdot\\
\cdot \quad \cdot \quad &\cdot\\
a_qx + b_qy &= k_q.
\end{aligned}
\tag{110}
$$

It is to be noted that $a_1 \neq 0$ or $b_1 \neq 0$, by the hypothesis in section 2. In general,

$$
a_i \neq 0 \quad \text{or} \quad b_i \neq 0 \quad (i = 1, \cdots, q). \tag{111}
$$

Now, if x and y are interpreted as rectangular coordinates in a plane, then the locus of the equation $ax + by = k$, in which $a \neq 0$ or $b \neq 0$, is a straight line. The numbers a, b, k in the equation of this line determine the direction of the line and a point on the line. Thus, if $b = 0$, then this line is perpendicular to the X-axis, and the point $(k/a, 0)$ is on the line. If $b \neq 0$, then the slope of this line is $-a/b$, and the point $(0, k/b)$ is on the line.

It will now be proved that the lines L_1 and L_2, whose equations are

$$
\begin{aligned}
a_1x + b_1y &= k_1,\\
a_2x + b_2y &= k_2,
\end{aligned}
\tag{112}
$$

are parallel if and only if there are constants p and q such that

$$
pa_1 = qa_2, \quad pb_1 = qb_2, \quad \text{and} \quad p \neq 0 \quad \text{or} \quad q \neq 0. \tag{113}
$$

If (113) hold and if $p = 0$, then $q \neq 0$, $a_2 = 0$, $b_2 = 0$, and hence (111) is contradicted. Therefore (113) and (111) imply $p \neq 0$. In the same way they imply $q \neq 0$. Now, if $b_1 = 0$ in (113), it follows that $b_2 = 0$, $a_1 \neq 0$, $a_2 \neq 0$. Then L_1 and L_2 are perpendicular to the X-axis and hence are parallel. Again, if $b_1 \neq 0$ in (113), then $b_2 \neq 0$. Then the slope of L_1 is $-a_1/b_1$, and the slope of L_2 is $-a_2/b_2$. By (113) these slopes are equal, and L_1 is parallel to L_2.

Next it will be proved that, if L_1 is parallel to L_2, then there are constants p and q such that (113) hold. First, if $b_1 = 0$, then L_1 is perpendicular to the X-axis. Therefore L_2 is perpendicular to the X-axis, and $b_2 = 0$. By (111) $a_1 \neq 0$ and $a_2 \neq 0$. If q and p are defined to be a_1 and a_2 respectively, then (113) hold.

Again if $b_1 \neq 0$ then L_1 is not perpendicular to the X axis Therefore L_2 is not perpendicular to the X axis and $b_2 \neq 0$ The slopes $-a_1/b_1$ and $-a_2/b_2$ are equal since the lines are parallel Hence $b_2a_1 - b_1a_2$ Therefore (113) hold if q and p are defined to be b_1 and b_2 respectively

The statement that the set $a_1\ b_1$ is proportional to the set $a_2\ b_2$ means by definition that (113) hold Therefore parallelism of two lines is equivalent to the fact that the set of coefficients of the variables in the equation of one line is proportional to the set of coefficients of the variables in the equation of the other line The fact that the set $a_1\ b_1$ is proportional to the set $a_2\ b_2$ is also written in the form

$$(114) \qquad a_1\ b_1 = a_2\ b_2$$

It will now be proved that there are constants p and q such that (113) are true if and only if

$$(115) \qquad r = 1$$

for the c m of (112) First if (113) are true then $p \neq 0$ and

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \frac{1}{p}\begin{vmatrix} pa_1 & pb_1 \\ a_2 & b_2 \end{vmatrix} = \frac{1}{p}\begin{vmatrix} qa_2 & qb_2 \\ a_2 & b_2 \end{vmatrix} = \frac{q}{p}\begin{vmatrix} a_2 & b_2 \\ a_2 & b_2 \end{vmatrix} = 0$$

Therefore $r = 1$ Next if $r = 1$ then $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ and $a_1b_2 = a_2b_1$ If $b \neq 0$ and p and q are defined to be b_2 and b_1 respectively then (113) hold If $b_1 = 0$ then $a_1 \neq 0$ Also then (113) hold if p and q are defined to be a_2 and a_1 respectively This completes the proof of the equivalence of (113) and (115) Therefore parallelism of the lines (112) is equivalent to (115)

It will now be proved that if p and q are constants such that (113) hold and if also

$$(116) \qquad pk_1 = qk_2$$

then L_1 and L_2 are coincident It has already been proved that (113) imply $pq \neq 0$ Therefore the equation of L_1 can be written in the form $pa_1x + pb_1y - pk_1$ and hence by (113) and (116) in the form $qa_2x + qb_2y = qk_2$ Since $q \neq 0$ this gives the equation of L_2 This proof also shows that (113) and

$$(117) \qquad pk_1 \neq qk_2$$

hold if and only if L_1 and L_2 are distinct parallel lines

By the methods which were used in the proof of the equivalence of (113) and (115) it can be proved that conditions (113) and (116) are equivalent to $r = 1 = r_a$, and that conditions (113) and (117) are equivalent to $r = 1$, $r_a = 2$. This completes the proof of the last two sentences in theorem 9.

Therefore two coincident lines illustrate geometrically theorem 6 if $n = 2 = q$, $r = 1 = r_a$. Two distinct parallel lines illustrate geometrically theorem 6 if $n = 2 = q$, $r = 1$, $r_a = 2$.

It will now be proved that, if $r = 2 = r_a$, then L_1 and L_2 intersect in one and only one point. By theorem 6 and the hypothesis that $r = 2 = r_a$, there is one and only one solution of equations (112). If s and t are the values of x and y which constitute this solution, then the point (s, t) is on L_1 and on L_2, and it is the only point on L_1 and on L_2.

Conversely, if L_1 and L_2 intersect in a unique point, there is one and only one solution of (112). Hence $r = 2 = r_a$ by theorem 6.

This completes the proof of theorem 9.

THEOREM 9. *The rank of the coefficient matrix of two linear equations in two variables is designated by r, and the rank of the augmented matrix by r_a. The two lines which are the loci of these equations intersect in one and only one point if and only if $r = 2 = r_a$. They are distinct parallel lines if and only if $r = 1$, $r_a = 2$. They are coincident if and only if $r = 1 = r_a$.*

The a.m. of the lines whose equations are

$$\begin{aligned} a_1x + b_1y &= k_1, \\ a_2x + b_2y &= k_2, \qquad (118) \\ a_3x + b_3y &= k_3 \end{aligned}$$

is

$$\begin{bmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{bmatrix}.$$

Since the c.m. has only two columns, therefore $r \leqq 2$. But r_a may be 3. If $r = 2$ and $r_a = 3$, then the three lines have no point in common, because, by theorem 6, equations (118) have no solution. However, there are two of equations (118) whose c.m. is of

rank 2 If theorem 9 is applied to the three pairs of equations in (118), it is found that

(i) the three lines intersect in three non-collinear distinct points and determine a triangle, or
(ii) two of the three lines intersect in one and only one point, and the third line is parallel to one of these two lines and not coincident with it

If $r = 2 = r_a$, then the three lines have one and only one point in common, by theorem 6 Two of these lines have a coefficient matrix of rank 2 and determine a pencil of lines The third line is a line of this pencil The third line may coincide with one of the two lines which determine the pencil

If $r = 1$, $r_a = 2$ then the three lines have no point in common If theorem 9 is applied to the three pairs of equations in (118), it is found that

(iii) the three lines are parallel and no two are coincident, or
(iv) two of the lines are distinct parallel lines and the third line coincides with one of these two lines

If $r = 1 = r_a$, then the three lines are coincident by theorem 9 applied to the three pairs of lines

This completes the proof of theorem 10

THEOREM 10 *The rank of the coefficient matrix of three linear equations in two variables is designated by r, and the rank of the augmented matrix by r_a Then $r \leq 2$, and $r_a \leq 3$ The conditions on r and r_a in the following table are necessary and sufficient for the corresponding geometric relation*

r	r_a	Geometric relation
2	3	(i) or (ii)
2	2	unique common point
1	2	(iii) or (iv)
1	1	coincident lines

If $q > 3$ in (110) then $r \leq 2$ and $r_a \leq 3$ Therefore the four pairs of values of r and r_a in theorem 10 are the only possibilities Then corresponding geometric relations can be proved by the methods which were used in the proof of theorem 10

If (110) are homogeneous equations, that is, if $k_1 = 0$, $\cdots$, $k_q = 0$, then they illustrate (85) if $n = 2$ The lines all pass

through the origin. Theorem 7 is illustrated if $r = 1$. Then $r_a = 1$, and all the lines are coincident. Since $n - r = 1$, therefore the geometrical interpretation of theorem 8 is that, if (s, t) is a point on this line, then all points on this line are obtained from (ms, mt) by assigning all real values to m.

If $n = 3$, equations (26) may be written more simply. Thus, if x_1, x_2, x_3 are replaced by x, y, z respectively, and if a_{i1} is replaced by a_i, a_{i2} by b_i, a_{i3} by c_i, then these equations become

$$
\begin{aligned}
a_1x + b_1y + c_1z &= k_1,\\
a_2x + b_2y + c_2z &= k_2,\\
\cdot \quad \cdot \quad \cdot \quad & \quad \cdot\\
\cdot \quad \cdot \quad \cdot \quad & \quad \cdot\\
\cdot \quad \cdot \quad \cdot \quad & \quad \cdot\\
a_qx + b_qy + c_qz &= k_q.
\end{aligned} \tag{119}
$$

It is to be noted that

$$a_i \neq 0 \quad \text{or} \quad b_i \neq 0 \quad \text{or} \quad c_i \neq 0 \quad (i = 1, \cdots, q). \tag{120}$$

Similarly at least one of $a_1, \cdots, a_q$ is not zero, at least one of $b_1, \cdots, b_q$ is not zero, and at least one of $c_1, \cdots, c_q$ is not zero.

If x, y, z are interpreted as rectangular coordinates in space of three dimensions, then the locus of the equation $ax + by + cz = k$, in which $a \neq 0$ or $b \neq 0$ or $c \neq 0$, is a plane. It is explained in solid analytic geometry percisely how the numbers a, b, c in the equation of this plane determine the direction of the line which can be drawn through the origin perpendicular to the plane. This line is called the normal to the plane from the origin. It is also explained how a, b, c, k determine the distance from the origin to the point of intersection of this normal and the plane. This distance is the perpendicular distance from the origin to the plane.

By the use of these facts from solid analytic geometry it can be proved that the planes whose equations are

$$
\begin{aligned}
a_1x + b_1y + c_1z &= k_1,\\
a_2x + b_2y + c_2z &= k_2,
\end{aligned} \tag{121}
$$

are parallel if and only if there are constants p and q such that

$$pa_1 = qa_2, \quad pb_1 = qb_2, \quad pc_1 = qc_2, \tag{122}$$

$$\text{and} \quad p \neq 0 \quad \text{or} \quad q \neq 0.$$

The statement that the set a_1 b_1 c_1 is proportional to the set a_2 b_2 c_2 means by definition that (122) hold Therefore parallelism of two planes is equivalent to the fact that the set of the coefficients of the variables in the equation of one plane is proportional to the set of the coefficients of the variables in the equation of the other plane The fact that the set a_1 b_1 c_1 is proportional to the set a_2 b_2 c_2 is also written in the form

$$(123) \qquad a_1\ b_1\ c_1 - a_2\ b_2\ c_2$$

By the methods used in the proof of the equivalence of (113) and (115) it can be proved that (122) are true if and only if

$$(124) \qquad r = 1$$

for the c m of (121) Furthermore if (122) hold and if also

$$(125) \qquad pk_1 = qk_2$$

then the planes are at the same distance from the origin in the same direction along the normal If (122) hold and if

$$(126) \qquad pk_1 \neq qk_2$$

then the planes are at different distances or they are at the same distance measured in opposite directions along the normal Therefore the planes are coincident if (122) and (125) hold whereas they are parallel and distinct if (122) and (126) hold Now (122) and (125) are equivalent to $r = 1 = r_a$ Also (122) and (126) are equivalent to $r = 1$ $r_a - 2$ This completes the proof of the last two sentences of theorem 11

Therefore two coincident planes illustrate geometrically theorem 6 if n 3 q 2 $r = 1 = r_a$ Two distinct parallel planes illustrate geometrically theorem 6 if $n = 3$ $q - 2$ r 1 $r_a = 2$

Now by (120) it is known that $r = 1$ or 2 in the c m of (121) Also if $r = 2$ then r_a 2 for (121) By theorem 6 if $n = 3$ q 2 there is a single infinity of solutions of (121) if and only if $r = 2 = r_a$ The planes intersect in a line This completes the proof of theorem 11

THEOREM 11 *The rank of the coefficient matrix of two linear equations in three variables is designated by r and the rank of the augmented matrix by r_a The two planes which are the loci of these*

equations intersect in a straight line if and only if $r = 2 = r_a$. They are parallel and distinct if and only if $r = 1$, $r_a = 2$. They are coincident if and only if $r = 1 = r_a$.

All the possible relations between three planes will now be characterized by conditions on the rank r of the c.m., and the rank r_a of the a.m., of their equations

$$
\begin{aligned}
a_1x + b_1y + c_1z &= k_1,\\
a_2x + b_2y + c_2z &= k_2, \qquad (127)\\
a_3x + b_3y + c_3z &= k_3.
\end{aligned}
$$

If $r = 3 = r_a$, then, by theorem 6, there is a unique solution of equations (127). Then the planes have one and only one point in common and determine a trihedral angle. If $r = 2$, $r_a = 3$, then, by theorem 6, the equations have no solution and the planes have no point in common. However, there are two of the equations whose c.m. is of rank 2. If theorem 11 is applied to the three pairs of equations in (127), it is found that

(i) the three planes intersect in three parallel lines and form a triangular prismatic surface; or
(ii) two of the planes intersect in a line, and the third plane is parallel to one of these two planes.

If $r = 2 = r_a$, then the equations have a single infinity of solutions, and the three planes have a line in common. The three planes are in the pencil of planes determined by a pair of them. If $r = 1$, $r_a = 2$, then the planes have no point in common. If theorem 11 is applied to the three pairs of equations in (127), it is found that

(iii) the planes are parallel and distinct; or
(iv) two of the planes are parallel and distinct, and the third plane coincides with one of these two planes.

If $r = 1 = r_a$, then the three planes are coincident. This completes the proof of theorem 12.

THEOREM 12. *The rank of the coefficient matrix of three linear equations in three variables is designated by r, and the rank of the augmented matrix by r_a. The conditions on r and r_a in the following*

table are necessary and sufficient for the corresponding geometric relation

r	r_a	Geometric relation
3	3	unique common point
2	3	(i) or (ii)
2	2	unique common line
1	2	(iii) or (iv)
1	1	coincident planes

By the preceding methods all possible relations between four planes can be characterized by conditions on the rank r of the c m and the rank r_a of the a m of their equations It is to be noted that $r \leq 3$ and $r_a \leq 4$ If $r = 3$ and $r_a = 4$ the planes have no point in common However there are three equations whose c m is of rank 3 These three planes have a unique point in common If theorem 11 is applied in turn to the fourth plane and each of these three planes it is found that the four planes intersect in four distinct points and form a tetrahedron or the fourth plane is parallel to one of the three planes and has a unique *point in common with the other two planes* The *other possible* relations between r and r_a and the corresponding geometric relations of the planes can be determined by these methods These same methods can be used if there are more than four planes

If (119) are homogeneous equations that is if $k_1 = 0$ $k_q = 0$ then they illustrate (85) if $n = 3$ The planes all pass through the origin The geometric interpretation of theorem 7 with $r = 2$ is a set of planes which have in common a line through the origin By theorem 8 if $(s\ t\ u)$ is a point on this line then all points on this line are obtained from $(ms\ mt\ mu)$ by assigning all real values to m The geometric interpretation of theorem 7 with $r = 1$ is a set of coincident planes passing through the origin By theorem 8 if $(s_1\ t_1\ u_1)$ and $(s_2\ t_2\ u_2)$ are two points on this plane such that the line joining them does not pass through the origin then all points on this plane are obtained from $(m_1s_1 + m_2s_2\ \ m_1t_1 + m_2t_2\ \ m_1u_1 + m_2u_2)$ by assigning all real values to m_1 and independently all real values to m_2

PROBLEMS

Find the geometric relations of the lines in the following systems

1 $2x + 3y = 7$
$\quad x + 5y = -1$
$\quad 3x - y = 2$

2 $4x + 3y = 5$
$\quad x - 2y = 6$
$\quad 2x + y = 1$

3. $$\begin{aligned} x - 2y &= 3, \\ 3x + y &= 5, \\ 13x + 9y &= 19, \\ x + y &= 1. \end{aligned}$$

4. $$\begin{aligned} 2x + y &= 4, \\ x - 3y &= 1, \\ 5x + 2y &= 7, \\ 4x + 9y &= 10. \end{aligned}$$

5. $$\begin{aligned} 3x + 2y &= 4, \\ x - 7y &= 1, \\ x + 39y &= 3, \\ 7x + 20y &= 10. \end{aligned}$$

6. $$\begin{aligned} x - 2y &= 1, \\ 2x + 5y &= 3, \\ 4x + 19y &= 7, \\ 2x - 13y &= 1. \end{aligned}$$

Find the geometric relations of the systems of planes whose equations are in the problems on p. 112.

CHAPTER 8

COMPLEX NUMBERS AND THE FUNDAMENTAL THEOREM OF ALGEBRA

1 **Complex numbers** The real numbers 2 and 3 determine two *ordered pairs* of numbers These pairs are given the notations $(2, 3)$ and $(3, 2)$ If a, b, c, d are real numbers, the statement that $(a, b) = (c, d)$ means that $a = c$ and $b = d$

The symbol $(2\ 3) + (5, -4)$ means, by definition, the pair $(2 + 5\ 3 - 4)$, that is the pair $(7, -1)$ If a, b, c, d are real numbers, then by definition,

$$(1) \qquad (a\ b) + (c, d) = (a + c, b + d)$$

One of the properties of real numbers is that addition is commutative This means that if a, b, c d are real numbers, then $a + c = c + a$ and $b + d = d + b$ Therefore $(a + c, b + d) = (c + a, d + b)$ Hence $(a\ b) + (c, d) = (c, d) + (a, b)$ Therefore *addition of pairs is commutative* Again, one of the properties of real numbers is that addition is associative This means that, if a, b, c, d, e, f are real numbers, then $[a + c] + e = a + [c + e]$ and that $[b + d] + f = b + [d + f]$ It follows that $([a + c] + e, [b + d] + f) = (a + [c + e]\ b + [d + f])$ By the definition of addition of pairs $(a + c, b + d) + (e, f) = ([a + c] + e, [b + d] + f)$ Therefore, by (1), $[(a\ b) + (c, d)] + (e, f) = ([a + c] + e, [b + d] + f)$ Similarly $(a, b) + [(c, d) + (e, f)] = (a + [c + e], b + [d + f])$ Hence $[(a\ b) + (c\ d)] + (e, f) = (a, b) + [(c, d) + (e, f)]$ Therefore *addition of pairs is associative* In the set of real numbers zero has the property that, if a and c are real numbers, then $a + 0 = a$, and $0 + c = c$ Therefore the pair $(0, 0)$ has the property that $(a, b) + (0, 0) = (a, b)$ and $(0, 0) + (c, d) = (c, d)$ Hence $(0\ 0)$ is called *the zero pair* or *the zero in the set of all pairs*

The symbol $(a, b)(c\ d)$ is defined by

$$(2) \qquad (a\ b)(c, d) = (ac - bd, ad + bc)$$

For example, $(2, 3)(5, -4) = (10 + 12, -8 + 15) = (22, 7)$. Now multiplication of real numbers is associative and commutative. Also multiplication is distributive with respect to addition. This means that, if a, b, c are real numbers, then $a(b + c) = ab + ac$. All these properties of real numbers suffice to prove that *multiplication of pairs is associative, commutative, and distributive with respect to addition.* Subtraction, and division except by the zero pair, are defined as they are for real numbers.

An important property of real numbers is that, if p and q are real numbers such that $pq = 0$, then either $p = 0$ or $q = 0$. The analogous property of pairs is that, if $(a, b)(c, d) = (0, 0)$, then either $(a, b) = (0, 0)$ or $(c, d) = (0, 0)$. This fact can be proved by using properties of the set of real numbers. Other properties of number pairs can be proved from the properteis of real numbers.

The symbol S will designate the set of real numbers, and the symbol C the set of pairs of real numbers. These notations are used to suggest that real numbers are simple and that number pairs are somewhat complicated. It is to be noted especially that the rules which are used in manipulating these number pairs are either definitions or laws of operation which are proved by the use of these definitions and properties of the real number system, and that these laws are precisely the same laws as those used in operating with real numbers. These number pairs are also called numbers.

An illustration of a linear equation involving number pairs is

$$(5, 2)(x, y) = (-23, 14). \tag{3}$$

By (2) it is verified that $(-3, 4)$ is a solution of (3). An illustration of a quadratic equation involving number pairs is $(4, 1)(x, y)^2 + (1, -21)(x, y) + (-33, -4) = (0, 0)$. By (2), (1), and the laws of operation for C, it is verified that the pairs $(2, 3)$ and $(-1, 2)$ satisfy this equation. A quadratic equation to which reference will be made later is

$$(1, 0)(x, y)^2 + (-4, 0)(x, y) + (13, 0) = (0, 0). \tag{4}$$

The pairs $(2, 3)$ and $(2, -3)$ satisfy this equation.

There is an important subset T of the set C of pairs. By definition T is the set of all pairs $(a, 0)$. Now the real number b in S determines the pair $(b, 0)$ in T. Also the pair $(d, 0)$ in T is determined by the real number d in S. The notation $b \leftrightarrow (b, 0)$

is used to express these two facts It is said that $b \leftrightarrow (b\ 0)$ establishes *a one to-one correspondence of the set S and the set T*

If a and c are in S then $a + c$ is in S Also $a \leftrightarrow (a\ 0)$ $c \leftrightarrow (c\ 0)$ and $a + c \leftrightarrow (a + c\ 0)$ However by (1) $(a + c\ 0) = (a\ 0) + (c\ 0)$ This implies that if a and c are elements in S then the element in T which is obtained by first adding a and c in S and next finding the corresponding element in T is the same as the element in T which is obtained by first finding the corresponding elements $(a\ 0)$ and $(c\ 0)$ in T and next adding these elements in T This fact is also expressed in symbols by

$$(5) \qquad a + c \leftrightarrow (a\ 0) + (c\ 0)$$

This is an important property of the correspondence $b \leftrightarrow (b\ 0)$ This property is also expressed by the statement that *the correspondence $b \leftrightarrow (b\ 0)$ is preserved under addition* In the same way it is proved that

$$(6) \qquad ac \leftrightarrow (a\ 0)(c\ 0)$$

This property is also expressed by the statement that *the correspondence $b \leftrightarrow (b\ 0)$ is preserved under multiplication*

The statement that T *is isomorphic to* S means that there is a one-to-one correspondence of the set S to the set T and that this correspondence is preserved under addition and multiplication

Another notation for number pairs will now be explained If a and b are real numbers the symbol $a + bU$ is defined by

$$(7) \qquad a + bU = (a\ b)$$

It is to be noted especially that the $+$ on the left-hand side of (7) is not the $+$ used between real numbers and it is not the $+$ used between number pairs The $+$ and U on the left-hand side of (7) form a symbol to order the real numbers a and b in a way analogous to that in which the symbol ( ) orders them In the new notation (1) becomes

$$(8) \qquad [a + bU] + [c + dU] = [a + c] + [b + d]U$$

It is to be noted especially that the right-hand side of (8) is precisely the result which would have been obtained if the left-hand side had been rewritten by the rules of ordinary algebra In the new notation (2) becomes

$$(9) \qquad [a + bU][c + dU] - [ac - bd] + [ad + bc]U$$

If the left-hand side of (9) were rewritten by the rules of ordinary algebra, the result would be $ac + bdU^2 + (ad + bc)U$. Therefore the right-hand side can be obtained if the left-hand side is rewritten by the rules of ordinary algebra and the condition

$$U^2 = -1 \tag{10}$$

is used. The importance of this new notation for number pairs is the fact that addition and multiplication are performed as in ordinary algebra and results are simplified by (10).

If the symbol U is replaced by the symbol i, then (10), (8), and (9) become respectively

$$i^2 = -1, \tag{11}$$

$$[a + bi] + [c + di] = [a + c] + [b + d]i, \tag{12}$$

$$[a + bi][c + di] = [ac - bd] + [ad + bc]i. \tag{13}$$

These are the familiar rules for addition and multiplication of complex numbers. This completes the proof that *complex numbers are ordered pairs of real numbers. The subset T of the set C of ordered pairs is isomorphic to the set S of real numbers.* In this sense it may be said that the real numbers are a subset of the complex numbers. It is to be noted especially that number pairs are no less substantial and no more visionary than the numbers which are paired. Therefore it is inadvisable that these number pairs be called imaginary numbers, as they have been.

2. The fundamental theorem of algebra. If the notation

$$z = (x, y) \tag{14}$$

is used, then (4) becomes

$$(1, 0)z^2 + (-4, 0)z + (13, 0) = (0, 0). \tag{15}$$

By (5) and (6) each step in the usual process of completing the square to solve $z^2 - 4z + 13 = 0$ can also be taken to solve (15). Thus it is found that $[(1, 0)z - (2, 0) - (0, 3)][(1, 0)z - (2, 0) + (0, 3)] = (0, 0)$. Hence $z = (2, 3)$ or $z = (2, -3)$. In the same way it follows that the familiar quadratic formula of algebra is valid in solving equations involving number pairs.

The fundamental theorem of algebra states that there is at least one number (c, d) which satisfies the polynomial equation

$$(16)\quad (a_0, b_0)(x, y)^n + (a_1, b_1)(x, y)^{n-1} + \cdots + (a_{n-1}, b_{n-1})(x, y) + (a_n, b_n) = (0, 0)$$

In particular, there is at least one number (a, b) which satisfies the polynomial equation

$$(17)\quad (a_0, 0)(x, y)^n + (a_1, 0)(x, y)^{n-1} + \cdots + (a_{n-1}, 0)(x, y) + (a_n, 0) = (0, 0)$$

By (5) and (6) there is at least one complex number which satisfies the polynomial equation

$$(18)\qquad a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n = 0$$

in which the coefficients are real numbers.

Proofs of the fundamental theorem of algebra are given in the references cited at the end of this book. These proofs use properties of the set of real numbers, properties of the set of complex numbers, and properties of functions of a complex variable. This theorem will not be proved in this book.

CHAPTER 9

SYMMETRIC FUNCTIONS

1. Relation between the coefficients and the roots of a polynomial equation. If $x^2 + b_1x + b_2 = 0$ is an equation whose roots are r_1 and r_2, then, by theorem 15 of chapter 3,

$$(1) \qquad x^2 + b_1x + b_2 \equiv (x - r_1)(x - r_2).$$

Also, it is verified by performing the indicated operations that

$$(2) \qquad x^2 - (r_1 + r_2)x + r_1r_2 \equiv (x - r_1)(x - r_2).$$

Hence $(x - r_1)(x - r_2)$ is the factored form of each of the functions $x^2 + b_1x + b_2$ and $x^2 - (r_1 + r_2)x + r_1r_2$. This factored form may be used, as in section 2 of chapter 1, to compute functional values of these functions. Therefore, if s is any complex number, then $s^2 + b_1s + b_2 = s^2 - (r_1 + r_2)s + r_1r_2s$. Let s_2 and s_3 be complex numbers such that s, s_2, s_3 are all distinct. Then similar equations hold for s_2 and s_3. By theorem 7 of chapter 3 it follows that

$$(3) \qquad \begin{aligned} -b_1 &= r_1 + r_2, \\ b_2 &= r_1r_2. \end{aligned}$$

Again, if r_1, r_2, r_3 are the roots of

$$(4) \qquad x^3 + b_1x^2 + b_2x + b_3 = 0,$$

it is proved in the same way that

$$(5) \qquad \begin{aligned} -b_1 &= r_1 + r_2 + r_3, \\ b_2 &= r_1r_2 + r_1r_3 + r_2r_3, \\ -b_3 &= r_1r_2r_3. \end{aligned}$$

In the same way it is proved that, if

$$(6) \qquad x^4 + b_1x^3 + b_2x^2 + b_3x + b_4 = 0$$

is an equation whose roots are four complex numbers r_1 r_2 r_3 r_4 then

$$(7)\qquad \begin{aligned} -b_1 &= r_1 + r_2 + r_3 + r_4 \\ b_2 &= r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 \\ -b_3 &= r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 \\ b_4 &= r_1r_2r_3r_4 \end{aligned}$$

A notation will now be introduced by which (3_1) (5_1) and (7_1) can be expressed in a single statement If n is a positive integer and r_1 r_n are complex numbers then S_1 is defined by

$$(8)\qquad S_1 = r_1 + \quad + r_n$$

It is especially to be noted that it is not assumed that $r_1,$ $, r_n$ are all distinct The subscript 1 on S is used because $r_1 + \quad + r_n$ is linear in r_1 r_n Later when a sum of more complex numbers is to be used simultaneously with $r_1 + \quad + r_n$ the symbol T_1 *will be used to designate this second sum instead of indicating* by a second subscript on S the number of summands to which the symbol refers By (8) the equation

$$(9)\qquad S_1 = -b_1$$

becomes (3_1) if $n = 2$ (5_1) if $n = 3$ (7_1) if $n = 4$

A notation will now be introduced by which (3_2) (5_2) and (7_2) can be expressed in a single statement If n is an integer which is greater than 2 and if r_1 r_n is a set of complex numbers then $r_1r_2 = r_2r_1$ However there are sets of numbers for which $r_1r_2 \neq r_1r_3$ Hence for all sets of complex numbers r_1 $, r_n$ the products r_1r_2 and r_1r_3 are said to be distinct and the products r_1r_2 and r_2r_1 are said to be not distinct Therefore *the distinct products of the numbers r_1 r_n taken two at a time* are r_1r_2 , r_1r_n r_2r_3 r_2r_n $r_{n-1}r_n$ The sum of these distinct products is designated by S_2 Therefore

$$(10)\qquad S_2 = r_1r_2 + \quad + r_1r_n + r_2r_3 + \quad + r_2r_n + \quad + r_{n-1}r_n$$

If $n = 2$ then the sum on the right-hand side of (10) is interpreted to contain merely the one term r_1r_2 Hence S_2 is defined if n is an integer which is greater than 1 By this definition the equation

$$(11)\qquad S_2 = b_2$$

becomes (3_2) if $n = 2$ (5_2) if $n = 3,$ (7_2) if $n = 4$

Similarly, if n is an integer which is greater than 3, and if $r_1, \cdots, r_n$ is a set of complex numbers, then, by definition, $r_1r_2r_3$ and $r_1r_2r_4$ are distinct products, but $r_1r_2r_3$ and $r_1r_3r_2$ are not distinct products. If $n = 4$, the distinct products of r_1, r_2, r_3, r_4 taken three at a time are $r_1r_2r_3, r_1r_2r_4, r_1r_3r_4, r_2r_3r_4$. If $n = 3$, then there is only one product of r_1, r_2, r_3 taken three at a time, namely, $r_1r_2r_3$. In general, if n is an integer which is greater than 2, then S_3 is, by definition, the sum of the distinct products of $r_1, \cdots, r_n$ taken three at a time. By this definition the equation

$$S_3 = -b_3 \tag{12}$$

becomes (5_3) if $n = 3$, (7_3) if $n = 4$.

In general, if k and n are positive integers such that $1 \leqq k \leqq n$, and if $r_1, \cdots, r_n$ are n complex numbers, then *a product of* $r_1, \cdots, r_n$ *taken* k *at a time* is, by definition, a product formed from k of these numbers with all the subscripts different. *Two such products are distinct*, by definition, if and only if the subscripts of the second product do not form a rearrangement of the subscripts of the first product. Then, by definition,

(13) S_k is the sum of the distinct products of

$r_1, \cdots, r_n$ taken k at a time.

If $k = 1$, then (13) is interpreted to mean (8). By this definition the equation

$$S_4 = b_4 \tag{14}$$

becomes (7_4) if $n = 4$. Also

$$S_k = (-1)^k b_k, \tag{15}$$

becomes (9), (11), (12), (14) if $k = 1, 2, 3, 4$ respectively. Since this equation has been proved if $n = 2, 3, 4$ and $1 \leqq k \leqq n$, therefore theorem 1 has been verified for $n = 2, 3, 4$.

THEOREM 1. *If n is an integer which is greater than 1, if $r_1, \cdots, r_n$ are complex numbers, if*

$$x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n = 0 \tag{16}$$

is an equation whose leading coefficient is 1 and whose roots are $r_1, \cdots, r_n$, and if S_k is defined by (13), then

$$S_k = (-1)^k b_k, \quad 1 \leqq k \leqq n. \tag{17}$$

Theorem 1 will be proved by mathematical induction The lemma for the induction remains to be proved This lemma states that if n_0 is a value of n for which the theorem is true, then $n_0 + 1$ is a value of n for which the theorem is true This lemma will now be proved It is given that $r_1, \cdots, r_{n_0+1}$ are complex numbers and that

$$(18) \qquad x^{n_0+1} + d_1 x^{n_0} + \cdots + d_{n_0} x + d_{n_0+1} = 0$$

is an equation whose roots are $r_1, \cdots, r_{n_0+1}$ By definition,

(19) T_j is the sum of the distinct products of

$$r_1, \cdots, r_{n_0+1} \text{ taken } j \text{ at a time, } 1 \leqq j \leqq n_0 + 1$$

It is to be proved that

$$(20) \qquad T_j = (-1)^j d_j, \quad 1 \leqq j \leqq n_0 + 1$$

By definition S_k is the sum in (13) with n replaced by n_0 If the expanded form of $(x - r_1) \cdots (x - r_{n_0})$ is given the notation $x^{n_0} + c_1 x^{n_0-1} + \cdots + c_{n_0-1}x + c_{n_0}$ then

$$(21) \quad (x - r_1) \cdots (x - r_{n_0}) \equiv x^{n_0} + c_1 x^{n_0-1} + \cdots + c_{n_0-1}x + c_{n_0}$$

Now by the hypothesis of the lemma for the induction,

$$(22) \qquad S_k = (-1)^k c_k, \quad 1 \leqq k \leqq n_0$$

Therefore

$$(23) \quad (x - r_1) \cdots (x - r_{n_0}) \equiv x^{n_0} - S_1 x^{n_0-1} + \cdots + (-1)^{n_0-1} S_{n_0-1} x + (-1)^{n_0} S_{n_0}$$

The result of multiplication of both sides of (23) by $x - r_{n_0+1}$ will not be exhibited If the operations indicated on the right-hand side of this result are performed, and if all terms involving like powers of x are combined, the result is

$$\begin{aligned}(24) \quad (x - r_1) \cdots (x - r_{n_0})(x - r_{n_0+1}) &\equiv x^{n_0+1} + (-S_1 - r_{n_0+1})x^{n_0} + (S_2 + S_1 r_{n_0+1})x^{n_0-1} \\ &\quad + \cdots + [(-1)^{n_0} S_{n_0} + (-1)^{n_0-1} S_{n_0-1}(-r_{n_0+1})]x \\ &\quad + (-1)^{n_0} S_{n_0}(-r_{n_0+1})\end{aligned}$$

By the definitions of S_1 and T_1 it follows that

$$(25) \qquad S_1 + r_{n_0+1} = T_1$$

Again, by the definitions

$$S_{n_0} r_{n_0+1} = T_{n_0+1}. \tag{26}$$

On the right-hand side of (24) all the terms except the first, second, and last can be written simultaneously by

$$(-1)^k (S_k + S_{k-1} r_{n_0+1}) x^{n_0+1-k}. \tag{27}$$

It will now be proved that

$$S_k + S_{k-1} r_{n_0+1} = T_k, \quad 2 \leqq k \leqq n. \tag{28}$$

By the definitions of S_2, S_1, and T_2 it is found that

$$S_2 + S_1 r_{n_0+1} = T_2. \tag{29}$$

In general, by (13), if $k > 1$ then each term in S_{k-1} is a product of $k - 1$ factors with distinct subscripts from $r_1, \cdots, r_{n_0}$. Therefore each term in the expanded form of $S_{k-1} r_{n_0+1}$ is a product of k distinct factors from $r_1, \cdots, r_{n_0+1}$. Therefore, by (19), each term in the expanded form of $S_{k-1} r_{n_0+1}$ is a term in T_k. Again, by (13) and (19), each term in S_k is a term in T_k. Moreover the terms in S_k and the terms in the expanded form of $S_{k-1} r_{n_0+1}$ are all distinct. Therefore each term in $S_k + S_{k-1} r_{n_0+1}$ is a term in T_k. The converse of this statement will now be proved. Each term in T_k either involves r_{n_0+1} or it does not involve r_{n_0+1}. If it does not involve r_{n_0+1} this term in T_k is a term in S_k. If it does involve r_{n_0+1} this term in T_k is a term in the expanded form of $S_{k-1} r_{n_0+1}$. Also the terms in T_k are all distinct. This completes the proof of (28).

Substitution of (25), (26), and (28) in (24) gives

$$(x - r_1) \cdots (x - r_{n_0+1}) \equiv x^{n_0+1} - T_1 x^{n_0} + T_2 x^{n_0-1} + \cdots + (-1)^{n_0} T_{n_0} x + (-1)^{n_0+1} T_{n_0+1}. \tag{30}$$

By the hypothesis that $r_1, \cdots, r_{n_0+1}$ are the roots of (18) and by theorem 15 of chapter 3 it follows that the left-hand side of (18) is identically equal to the right-hand side of (30). Therefore the coefficients of like powers of x are equal, and (20) hold. This completes the proof of theorem 1.

It is to be noted especially that the leading coefficient in (16) is 1.

2 The fundamental theorem on symmetric functions In (13) there appear certain simple expressions involving the roots $r_1, \cdots, r_n$ of an equation These same expressions in n independent variables $x_1, \cdots, x_n$ are designated by $E_1, \cdots, E_n$ Thus, in the definition preceding (13), $r_1, \cdots, r_n$ are replaced by $x_1, \cdots, x_n$ Also, if k is an integer such that $1 \leq k \leq n$, then, by definition, E_k is the sum of the distinct products of $x_1, \cdots, x_n$ taken k at a time Therefore, in symbols, if $n > 2$ then

$$
\begin{aligned}
E_1 &\equiv x_1 + \cdots + x_n, \\
E_2 &\equiv x_1x_2 + \cdots + x_1x_n + x_2x_3 + \cdots + x_2x_n \\
&\quad + \cdots + x_{n-1}x_n, \\
&\cdots\cdots\cdots\cdots \\
E_n &\equiv x_1x_2 \cdots x_n
\end{aligned}
\tag{31}
$$

If $n = 2$ then $E_1 \equiv x_1 + x_2$ and $E_2 \equiv x_1x_2$ Also, by definition,

$$E_k(x_1, \cdots, x_n) \equiv E_k, \quad k = 1, \cdots, n \tag{32}$$

In these symbols, and later when dots occur in the symbol of a function, the dots indicate that the subscripts of the omitted variables are in natural order

An important property of each of the functions (31) will now be explained This property will be proved in detail if $n = 4$ and the function is E_1 By (32) and (31), if $n = 4$ then

$$E_1(x_1, x_2, x_3, x_4) \equiv x_1 + x_2 + x_3 + x_4 \tag{33}$$

Therefore $E_1(x_2, x_1, x_3, x_4) \equiv x_2 + x_1 + x_3 + x_4$ Also addition is commutative Therefore the function $E_1(x_2, x_1, x_3, x_4)$ is identically equal to the function $E_1(x_1, x_2, x_3, x_4)$, that is,

$$E_1(x_2, x_1, x_3, x_4) \equiv E_1 \tag{34}$$

In the same way it is proved that

$$E_1(x_3, x_2, x_1, x_4) \equiv E_1, \tag{35}$$

$$E_1(x_4, x_2, x_3, x_1) \equiv E_1, \tag{36}$$

$$E_1(x_1, x_3, x_2, x_4) \equiv E_1, \tag{37}$$

$$E_1(x_1, x_4, x_3, x_2) \equiv E_1, \tag{38}$$

$$E_1(x_1, x_2, x_4, x_3) \equiv E_1 \tag{39}$$

Identities (34) to (39) are summarized in the statement that E_1 is an illustration of a function $f(x_1, \cdots, x_4)$ which has the property that each of the functions of $x_1, \cdots, x_4$ which is obtained by interchanging two of $x_1, \cdots, x_4$ in $f(x_1, \cdots, x_4)$ is identically equal to $f(x_1, \cdots, x_4)$.

It will now be proved that

$$E_1(x_3, x_1, x_4, x_2) \equiv E_1. \tag{40}$$

Since $E_1(x_1, x_2, x_3, x_4)$ is a very simple function of $x_1, \cdots, x_4$, direct verification is the easiest method of proving (40). Another method of proving (40) will now be explained, because the proof by this method would be easy even if the function were complicated. This method is also important because it can be used to prove for any function many identities analogous to (40), after identities analogous to (34) ,$\cdots$, (39) have been proved for this function. Now, if $z_1, \cdots, z_4$ are arbitrary variables, by (39) it is true that $E_1(z_1, z_2, z_4, z_3) \equiv E_1(z_1, z_2, z_3, z_4)$. Replacement of z_1, z_2, z_3, z_4 by x_3, x_1, x_2, x_4 respectively shows that

$$E_1(x_3, x_1, x_4, x_2) \equiv E_1(x_3, x_1, x_2, x_4). \tag{41}$$

Similarly, by (37), $E_1(z_1, z_3, z_2, z_4) \equiv E_1(z_1, z_2, z_3, z_4)$. Replacement of z_1, z_2, z_3, z_4 by x_3, x_1, x_2, x_4 respectively shows that

$$E_1(x_3, x_2, x_1, x_4) \equiv E_1(x_3, x_1, x_2, x_4). \tag{42}$$

Similarly, by (35), $E_1(z_3, z_2, z_1, z_4) \equiv E_1(z_1, z_2, z_3, z_4)$. Replacement of z_1, z_2, z_3, z_4 by x_3, x_2, x_1, x_4 respectively shows that

$$E_1(x_1, x_2, x_3, x_4) \equiv E_1(x_3, x_2, x_1, x_4). \tag{43}$$

Then (40) follows from (41), (42), (43).

In general, by this method it is a corollary of (34) to (39) that, if $i_1i_2i_3i_4$ is an arrangement of 1, 2, 3, 4, then

$$E_1(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}) \equiv E_1. \tag{44}$$

Thus, as a corollary of the property of $E_1(x_1, x_2, x_3, x_4)$ which was stated as a summary of (34) to (39), it follows that E_1 is an illustration of a function $f(x_1, \cdots, x_4)$ which has a second property, namely, that each of the functions of $x_1, \cdots, x_4$ which is obtained by rearranging $x_1, \cdots, x_4$ in $f(x_1, x_2, x_3, x_4)$ is identically equal to $f(x_1, x_2, x_3, x_4)$. The statement that f is a symmetric function of x_1, x_2, x_3, x_4 means, by definition, that f has the second property.

Conversely if a function has the second property then it has the first property

It can be verified that E_2 E_3 and E_4 are symmetric functions of x_1 x_4

If n is an arbitrary positive integer the statement that $f(x_1 \quad x_n)$ *is a symmetric function of* x_1 x_n means by definition that each function of x_1 x_n which is obtained by rearranging the variables x_1 x_n in $f(x_1 \quad x_n)$ is identically equal to $f(x_1 \quad x_n)$ This is equivalent to the statement that each function of x_1 x_n which is obtained by interchanging two of x_1 x_n in $f(x_1 \quad x_n)$ is identically equal to $f(x_1 \quad x_n)$ The functions E_1 E_n are called *the elementary symmetric functions of* x_1 x_n

The function $x_1^2 + x_2^2 + x_3^2 + x_4^2$ is a symmetric function of x_1 x_2 x_3 x_4 The notation Σx_1^2 is used to designate this function If $n = 3$ then Σx_1^2 means $x_1^2 + x_2^2 + x_3^2$ If there are n variables then Σx_1^2 means $x_1^2 + \quad + x_n^2$ This function Σx_1^2 is called a Σ function If $n = 3$ the function $x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2$ is another illustration of a Σ function It is designated by $\Sigma x_1^2x_2$ If $n = 2$ then $\Sigma x_1^2x_2$ means $x_1^2x_2 + x_2^2x_1$ In general *a Σ function* is the sum of all the distinct terms that can be formed from the exhibited term by replacing the list of subscripts in that term with a list of distinct integers from 1 n Therefore a Σ-*function is a symmetric function* in x_1 x_n Finally it is to be noted that $E_1 = \Sigma x_1$ $E_2 = \Sigma x_1x_2$ $E_{n-1} = \Sigma x_1x_2 \quad x_{n-1}$ $E_n = \Sigma x_1 \quad x_n$

If b is independent of x_1 x_n and if each of k_1 k_n is a positive integer then $bx_1^{k_1} \quad x_n^{k_n}$ is by definition a polynomial in x_1 x_n which consists of just one term The degree of this term in x_1 x_n is $k_1 + \quad + k_n$ In general by definition *a polynomial in* x_1 x_n is a single term of this type or a sum of a finite number of such terms *The degree of the polynomial in* x_1 x_n is the largest degree $k_1 + \quad + k_n$ among all those terms in the polynomial for which the coefficient b is not zero The statement that *a polynomial in* x_1 x_n *is homogeneous in* x_1 x_n means by definition that the degree $k_1 + \quad + k_n$ is the same for all the *terms in the polynomial* Thus $x_1^2 + x_1x_2 + x_3^2$ is a homogeneous polynomial of degree 2 in x_1 x_2 x_3 and $x_1^3 + x_1^2x_2 + x_3^3$ is a homogeneous polynomial of degree 3 in x_1 x_2 x_3 Again $x_1^2 + x_1x_2 + x_3$ is a polynomial of degree 2 in

x_1, x_2, x_3, but it is not homogeneous in x_1, x_2, x_3. Again, Σx_1^{-1} is symmetric in $x_1, \cdots, x_n$, but it is not a polynomial in $x_1, \cdots, x_n$.

The fundamental theorem on symmetric functions is a theorem about symmetric polynomials in $x_1, \cdots, x_n$. This theorem will now be illustrated, using the polynomial g which is defined by

$$(45) \quad g(x_1. x_2, x_3) \equiv 3x_1^2 + 3x_2^2 + 3x_3^3 + 5x_1x_2 + 5x_1x_3 + 5x_2x_3.$$

First, by (31) with $n = 3$,

$$(46) \quad E_1 \equiv x_1 + x_2 + x_3, \quad E_2 \equiv x_1x_2 + x_1x_3 + x_2x_3, \quad E_3 \equiv x_1x_2x_3.$$

Next, by performing the indicated operations and simplifying the result, it is verified that $3(x_1 + x_2 + x_3)^2 - (x_1x_2 + x_1x_3 + x_2x_3) \equiv g(x_1, x_2, x_3)$. This is precisely the meaning of the statement that $g(x_1, x_2, x_3)$ is equal to $3E_1^2 - E_2$ identically in x_1, x_2, x_3. This identity is indicated by

$$(47) \qquad g(x_1, x_2, x_3) \equiv 3E_1^2 - E_2.$$

In general, if $f(x_1, \cdots, x_n)$ is a polynomial in $x_1, \cdots, x_n$, and if $F(E_1, \cdots, E_n)$ is a polynomial in $E_1, \cdots, E_n$, then $f(x_1, \cdots, x_n) \equiv F(E_1, \cdots, E_n)$ means, by definition, that the polynomial in $x_1, \cdots, x_n$, which is obtained from $F(E_1, \cdots, E_n)$ by substitution from (31), equals $f(x_1, \cdots, x_n)$ identically in $x_1, \cdots, x_n$. Therefore (47) illustrates that part of the fundamental theorem which states that, if $f(x_1, \cdots, x_n)$ is a symmetric polynomial in $x_1, \cdots, x_n$, then there is a polynomial $F(E_1, \cdots, E_n)$ in $E_1, \cdots, E_n$ such that $f(x_1, \cdots, x_n) \equiv F(E_1, \cdots, E_n)$.

The other part of the fundamental theorem is illustrated more satisfactorily using the polynomial f which is defined by

$$(48) \quad f(x_1, x_2, x_3) \equiv c(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + b(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2).$$

If $H(E_1, E_2, E_3, b, c)$ designates the polynomial $cE_2^2 + (b - 2c)E_1E_3$ in E_1, E_2, E_3, b, c, then, by (46), $H(E_1, E_2, E_3, b, c)$ becomes a polynomial $h(x_1, x_2, x_3, b, c)$ in x_1, x_2, x_3, b, c. By performing the indicated operations and combining terms it is verified that $h(x_1, x_2, x_3, b, c)$ equals the right-hand side of (48) identically in x_1, x_2, x_3. This is precisely the meaning of the statement that $f(x_1, x_2, x_3)$ is equal to $cE_2^2 + (b - 2c)E_1E_3$ identically

in x_1, x_2, x_3 This identity is indicated by

(49) $$f(x_1, x_2, x_3) \equiv cE_2^2 + (b - 2c)E_1E_3$$

In general, if f is a polynomial in $x_1, \cdots, x_n$, and if F is a polynomial in $E_1, \cdots, E_n$ and the coefficients of f, then the statement that f *equals* F *identically in* $x_1, \cdots, x_n$ means, by definition, that the polynomial in $x_1, \cdots, x_n$ and the coefficients of f, which is obtained by substitution from (31) in F, equals f identically in $x_1, \cdots, x_n$ In notation this statement is expressed by $f \equiv F$ Therefore (49) illustrates both parts of the fundamental theorem that, if f is a symmetric polynomial in $x_1, \cdots, x_n$, then there is a polynomial F in $E_1, \cdots, E_n$ and the coefficients of f, with integral coefficients, such that $f \equiv F$

The symmetric polynomial $c(x_1^2 + x_2^2 + x_3^2) + b(x_1 + x_2 + x_3)$ is not homogeneous in x_1, x_2, x_3 However, it is the sum of two polynomials each of which is homogeneous and symmetric in x_1, x_2, x_3, because it is the sum $c\Sigma x_1^2 + b\Sigma x_1$ In general, a symmetric polynomial in $x_1, \cdots, x_n$, which is not homogeneous in $x_1, \cdots, x_n$, is a sum of polynomials in $x_1, \cdots, x_n$, each of which is symmetric and homogeneous in $x_1, \cdots, x_n$ Therefore, if the fundamental theorem is proved for all polynomials which are symmetric and homogeneous in $x_1, \cdots, x_n$, it will follow that the fundamental theorem is true for all polynomials which are symmetric in $x_1, \cdots, x_n$ For example, since $c(x_1^2 + x_2^2 + x_3^2) \equiv cE_1^2 - 2cE_2$ and $b(x_1 + x_2 + x_3) \equiv bE_1$, the non-homogeneous symmetric polynomial $c\Sigma x_1^2 + b\Sigma x_1$ is equal to $cE_1^2 - 2cE_2 + bE_1$ identically in x_1, x_2, x_3

The polynomial $f(x_1, x_2, x_3)$, which is defined to be

(50) $$\begin{aligned} &c(x_1^4x_2^3x_3 + x_1^4x_2x_3^3 + x_1^3x_2x_3^4 + x_1^3x_2^4x_3 + x_1x_2^4x_3^3 \\ &+ x_1x_2^3x_3^4) + 2c(x_1^4x_2^2x_3^2 + x_1^2x_2^4x_3^2 + x_1^2x_2^2x_3^4) \\ &+ d(x_1^3x_2^3x_3^2 + x_1^3x_2^2x_3^3 + x_1^2x_2^3x_3^3), \end{aligned}$$

is symmetric in x_1, x_2, x_3 It can also be written

(51) $$c\Sigma x_1^4x_2^3x_3 + 2c\Sigma x_1^4x_2^2x_3^2 + d\Sigma x_1^3x_2^3x_3^2$$

The polynomial in E_1, E_2, E_3, c, d, to which $f(x_1, x_2, x_3)$ is equal identically in x_1, x_2, x_3, will now be found by a method which illustrates the method of proof of the fundamental theorem for symmetric homogeneous polynomials in $x_1, \cdots, x_n$

As a first step in finding this polynomial in E_1, E_2, E_3, c, d, (50) is written with its terms in the order

$$\begin{aligned} &cx_1^4x_2^3x_3 + 2cx_1^4x_2^2x_3^2 + cx_1^4x_2x_3^3 + cx_1^3x_2^4x_3 \\ (52)\qquad &+ dx_1^3x_2^3x_3^2 + dx_1^3x_2^2x_3^3 + cx_1^3x_2x_3^4 + 2cx_1^2x_2^4x_3^2 \\ &+ dx_1^2x_2^3x_3^3 + 2cx_1^2x_2^2x_3^4 + cx_1x_2^4x_3^3 + cx_1x_2^3x_3^4. \end{aligned}$$

The rule for ordering the terms will now be explained. First, the coefficients are disregarded in determining this order. Next, the term in which $x_1^{h_1}x_2^{h_2}x_3^{h_3}$ is a factor precedes the term in which $x_1^{k_1}x_2^{k_2}x_3^{k_3}$ is a factor if and only if one of (53) holds:

$$\begin{aligned} &\text{(i)}\quad h_1 > k_1, \\ (53)\qquad &\text{(ii)}\quad h_1 = k_1, \quad \text{and} \quad h_2 > k_2, \\ &\text{(iii)}\quad h_1 = k_1, \quad h_2 = k_2, \quad \text{and} \quad h_3 > k_3. \end{aligned}$$

This condition, that one and only one of (53) holds, is equivalent to the condition that the first of the differences $h_1 - k_1$, $h_2 - k_2$, $h_3 - k_3$ which is not zero is indeed positive. It should be verified that the terms in (52) have been ordered by this rule.

The term $cx_1^4x_2^3x_3$ in (52), which precedes every other term in (52) when the terms are ordered by the rule which has just been explained, is called the highest term in (52). Also, when one term precedes another term in (52), then the former term is said to be higher than the latter term.

The second step in finding the polynomial in E_1, E_2, E_3, c, d which is identically equal to the polynomial (51) involves a new relationship between homogeneous symmetric polynomials. This relationship will now be explained. Thus, if $g(x_1, x_2, x_3)$ and $G(x_1, x_2, x_3)$ are two polynomials which are homogeneous and symmetric in x_1, x_2, x_3, then the statement that g is higher than G means, by definition, either that the degree of g is greater than the degree of G, or that the degree of g is equal to the degree of G and that the highest term in g is higher than the highest term in G. This last condition means that, if the degrees of g and G are equal, and if $ax_1^{p_1}x_2^{p_2}x_3^{p_3}$ and $bx_1^{q_1}x_2^{q_2}x_3^{q_3}$ are the highest terms of g and G respectively, then g is higher than G if and only if the first one of $p_1 - q_1$, $p_2 - q_2$, $p_3 - q_3$ which is not zero is positive.

The rule (53) for ordering the terms in a symmetric homogeneous polynomial in x_1 x_2 x_3 can be extended to give a rule for ordering the terms in a symmetric homogeneous polynomial in $x_1 \cdots x_n$. Thus the term in which $x_1^{h_1} \cdots x_n^{h_n}$ is a factor precedes the term in which $x_1^{k_1} \cdots x_n^{k_n}$ is a factor if and only if

(54) the first one of the differences $h_1 - k_1, \cdots, h_n - k_n$

which is not zero is positive

Also by definition the highest term in a symmetric homogeneous polynomial in $x_1 \cdots x_n$ is the term which precedes every other term when the terms have been ordered by this rule. Again when one term precedes another term then the former term is said to be higher than the latter term. Finally if g and G are two symmetric homogeneous polynomials in $x_1 \cdots x_n$ then the statement that g is higher than G means by definition either that the degree of g is greater than the degree of G or that their degrees are equal and that if the highest term in g is $ax_1^{p_1} \cdots x_n^{p_n}$ and the highest term in G is $bx_1^{q_1} \cdots x_n^{q_n}$ then the first one of $p_1 - q_1 \cdots p_n - q_n$ which is not zero is positive

It will now be proved that if $bx_1^{h_1} x_2^{h_2} \cdots x_n^{h_n}$ is the highest term in a polynomial which is homogeneous and symmetric in $x_1 \cdots x_n$ then

$$(55) \qquad h_1 \geqq h_2 \geqq \cdots \geqq h_n$$

This will be done by proving that if $h_1 < h_2$ then there is a contradiction and also that if j is an integer such that $1 < j < n$ and $h_1 - h_2 = \cdots = h_j < h_{j+1}$ then there is a contradiction. By the definition of a symmetric function if $bx_1^{h_1} x_2^{h_2} x_3^{h_3} \cdots x_n^{h_n}$ is a term in a symmetric function then $bx_2^{h_1} x_1^{h_2} x_3^{h_3} \cdots x_n^{h_n}$ is also a term in this function. If $h_1 < h_2$ then the latter term is higher than the former term. Therefore if $bx_1^{h_1} x_2^{h_2} x_3^{h_3} \cdots x_n^{h_n}$ is the highest term in this function and if $h_1 < h_2$ then there is a contradiction. Similarly there is a contradiction if $h_1 = h_2 = \cdots = h_j < h_{j+1}$. The property stated in (55) is illustrated if $n = 3$ by the first term in (52)

The highest term in Σx_1 is x_1. The highest term in $\Sigma x_1 x_2$ is $x_1 x_2$. If $n = 3$ it can be verified that the highest term in $(\Sigma x_1)(\Sigma x_1 x_2)$ is the product $(x_1)(x_1 x_2)$ of the highest term x_1 in Σx_1 and the highest term $x_1 x_2$ in $\Sigma x_1 x_2$. Thus the following lemma

has been verified if $n = 3$ and the symmetric polynomials are the particular functions Σx_1 and $\Sigma x_1 x_2$.

LEMMA 1. *If f and g are symmetric homogeneous polynomials in $x_1, \cdots, x_n$, then fg is a symmetric homogeneous polynomial in $x_1, \cdots, x_n$, and the highest term in fg is the product of the highest term in f and the highest term in g.*

Lemma 1 will now be proved. First, fg is a symmetric homogeneous polynomial in $x_1, \cdots, x_n$, by the definitions and the hypotheses in the lemma. Next, if $x_1^{m_1} \cdots x_n^{m_n}$ is a factor of a term in fg, then there is a product $x_1^{s_1} \cdots x_n^{s_n}$ which is a factor of a term in f, and a product $x_1^{t_1} \cdots x_n^{t_n}$ which is a factor of a term in g, such that $m_1 = s_1 + t_1, \cdots, m_n = s_n + t_n$. Again, it will be proved that, if $bx_1^{p_1} \cdots x_n^{p_n}$ is the highest term in f and if $cx_1^{q_1} \cdots x_n^{q_n}$ is the highest term in g, then $bcx_1^{p_1+q_1} \cdots x_n^{p_n+q_n}$ is a term in fg. This will be done by showing that the product of $bx_1^{p_1} \cdots x_n^{p_n}$ in f and $cx_1^{q_1} \cdots x_n^{q_n}$ in g is the only product of a term in f and a term in g which has $x_1^{p_1+q_1} \cdots x_n^{p_n+q_n}$ as a factor. Thus, if $x_1^{u_1} \cdots x_n^{u_n}$ from f and $x_1^{v_1} \cdots x_n^{v_n}$ from g are such that $u_1 + v_1 = p_1 + q_1, \cdots, u_n + v_n = p_n + q_n$, then $(p_1 - u_1) + (q_1 - v_1) = 0, \cdots, (p_n - u_n) + (q_n - v_n) = 0$. In the first of these equations either $p_1 - u_1 > 0$ or $p_1 - u_1 = 0$, since $bx_1^{p_1} \cdots x_n^{p_n}$ is the highest term of f. If $p_1 - u_1 > 0$, then it follows that $q_1 - v_1 < 0$. This contradicts the hypothesis that $cx_1^{q_1} \cdots x_n^{q_n}$ is the highest term in g. Therefore $p_1 - u_1 = 0$, and $q_1 - v_1 = 0$. Thus, from the first equation it follows that $p_1 = u_1$ and $q_1 = v_1$. Similarly from the second equation it follows that $p_2 = u_2$ and $q_2 = v_2$. Repetition of this argument proves that $p_1 = u_1, \cdots, p_n = u_n$; $q_1 = v_1, \cdots, q_n = v_n$. This is precisely the statement which was to be proved.

It will now be proved that the term $bcx_1^{p_1+q_1} \cdots x_n^{p_n+q_n}$ is the highest term in fg. This will be done by showing that, if $x_1^{m_1} \cdots x_n^{m_n}$ is a factor of a term in fg which is not the term $bcx_1^{p_1+q_1} \cdots x_n^{p_n+q_n}$, then one of $p_1 + q_1 - m_1, \cdots, p_n + q_n - m_n$ is not zero, and the first one which is not zero is indeed positive. Now, as noted earlier, there is a product $x_1^{s_1} \cdots x_n^{s_n}$ from f, and there is a product $x_1^{t_1} \cdots x_n^{t_n}$ from g, such that $m_1 = s_1 + t_1, \cdots, m_n = s_n + t_n$. Therefore $p_j + q_j - m_j = (p_j - s_j) + (q_j - t_j)$ if $j = 1, \cdots, n$. If $p_1 - s_1 + q_1 - t_1 \neq 0$, then either (i) $p_1 - s_1 \neq 0$ and $q_1 - t_1 \neq 0$, or (ii) $p_1 - s_1 \neq 0$ and $q_1 - t_1 = 0$, or

(iii) $p_1 - s_1 = 0$ and $q_1 - t_1 \neq 0$. If (i) is true, then $p_1 - s_1 > 0$, because $bx_1^{p_1} \cdots x_n^{p_n}$ is the highest term in f. Similarly $q_1 - t_1 > 0$. Therefore $p_1 - s_1 + q_1 - t_1 > 0$. Again, if (ii) is true, then $p_1 - s_1 > 0$ and $q_1 - t_1 = 0$, and hence $p_1 - s_1 + q_1 - t_1 > 0$. Finally, if (iii) is true, then $p_1 - s_1 = 0$ and $q_1 - t_1 > 0$, and hence $p_1 - s_1 + q_1 - t_1 > 0$. If $p_1 - s_1 + q_1 - t_1 = 0$, then $p_1 - s_1 = 0$ and $q_1 - t_1 = 0$, and the argument is repeated on $p_2 - s_2 + q_2 - t_2$. Thus finally it is proved that either $p_1 = s_1, \cdots, p_n = s_n, q_1 = t_1, \cdots, q_n = t_n$, that is, $m_1 = p_1 + q_1, \cdots, m_n = p_n + q_n$, or one of $p_1 + q_1 - m_1, \cdots, p_n + q_n - m_n$ is not zero and the first one which is not zero is positive. This completes the proof of lemma 1.

The second step in finding the polynomial in E_1, E_2, E_3, c, d which is identically equal to the polynomial (51) will now be explained. First the highest term $cx_1^4x_2^3x_3$ in $f(x_1, x_2, x_3)$ is written down. Next the coefficient c and the exponents 4, 3, 1 on x_1, x_2, x_3 respectively, in this highest term are used to construct the expression

$$cE_1^{4-3}E_2^{3-1}E_3^{1} \tag{56}$$

The expression obtained by substitution of (46) in (56) is a symmetric homogeneous polynomial in x_1, x_2, x_3 which is of the same degree in x_1, x_2, x_3 as $f(x_1, x_2, x_3)$. Therefore if $g(x_1, x_2, x_3)$ is defined by

$$g(x_1, x_2, x_3) \equiv f(x_1, x_2, x_3) - cE_1^{4-3}E_2^{3-1}E_3^{1}, \tag{57}$$

then $g(x_1, x_2, x_3)$ is a symmetric homogeneous polynomial in x_1, x_2, x_3 which is identically zero or of the same degree in x_1, x_2, x_3 as $f(x_1, x_2, x_3)$.

Now by (46) the highest term in E_1 is x_1, the highest term in E_2 is x_1x_2, and the highest term in E_3 is $x_1x_2x_3$. Therefore the highest term in the expression which is obtained by substitution of (46) in (56) is by lemma 1 $cx_1^{4-3}(x_1x_2)^{3-1}(x_1x_2x_3)^1$, that is $cx_1^4x_2^3x_3$. But this is also the highest term in $f(x_1, x_2, x_3)$. Therefore a term having $x_1^4x_2^3x_3$ as a factor does not appear in the difference $f(x_1, x_2, x_3) - cE_1^{4-3}E_2^{3-1}E_3^1$, that is, by (57), in $g(x_1, x_2, x_3)$. Therefore by the definitions, $f(x_1, x_2, x_3)$ is higher than $g(x_1, x_2, x_3)$.

The third step in finding the polynomial in E_1, E_2, E_3, c, d, which is identically equal to the polynomial (51), will now be ex-

plained. By (57), $f(x_1, x_2, x_3) \equiv cE_1E_2^2E_3 + g(x_1, x_2, x_3)$. Therefore, if a polynomial in E_1, E_2, E_3, c, d is found, to which $g(x_1, x_2, x_3)$ is equal identically in x_1, x_2, x_3, then the required polynomial in E_1, E_2, E_3, c, d for $f(x_1, x_2, x_3)$ will be known. By use of (50) and (46) in (57) it is found that

$$g(x_1, x_2, x_3) \equiv (d - 5c)\Sigma x_1^3x_2^3x_3^2. \tag{58}$$

The first step in finding a polynomial in E_1, E_2, E_3, c, d which is identically equal to $g(x_1, x_2, x_3)$ is exhibiting the highest term

$$(d - 5c)x_1^3x_2^3x_3^2 \tag{59}$$

in $g(x_1, x_2, x_3)$. The second step in finding this polynomial is using the coefficient $d - 5c$, and the exponents 3, 3, 2 of x_1, x_2, x_3 respectively, in this highest term to construct the expression

$$(d - 5c)E_1^{3-3}E_2^{3-2}E_3^2. \tag{60}$$

If $h(x_1, x_2, x_3)$ is defined by

$$h(x_1, x_2, x_3) \equiv g(x_1, x_2, x_3) - (d - 5c)E_1^{3-3}E_2^{3-2}E_3^2, \tag{61}$$

then $h(x_1, x_2, x_3)$ is a symmetric homogeneous polynomial in x_1, x_2, x_3 which is identically zero or of the same degree in x_1, x_2, x_3 as $g(x_1, x_2, x_3)$. By (46) and lemma 1, the highest term in (60) is $(d - 5c)x_1^{3-3}(x_1x_2)^{3-2}(x_1x_2x_3)^2$, that is, $(d - 5c)x_1^3x_2^3x_3^2$. This is also the highest term (59) in $g(x_1, x_2, x_3)$. Therefore, by (61), $g(x_1, x_2, x_3)$ is higher than $h(x_1, x_2, x_3)$.

By using (58) and (46) in (61), it is found that

$$h(x_1, x_2, x_3) \equiv 0. \tag{62}$$

Therefore, by (61) and (57),

$$f(x_1, x_2, x_3) \equiv cE_1E_2^2E_3 + (d - 5c)E_2E_3^2. \tag{63}$$

The expression on the right-hand side of (63) is the polynomial in E_1, E_2, E_3, c, d which is identically equal to (51). It is to be noted that the coefficients of this polynomial in E_1, E_2, E_3, c, d are integers.

In other particular illustrations the process would terminate at (57) if in (58) it were found that $g(x_1, x_2, x_3) \equiv 0$, but it would extend beyond (61) if in (62) it were found that $h(x_1, x_2, x_3)$ were

not identically zero. In all cases the process terminates after a finite number of steps, because the ordered triples of non-negative integers, which are exponents of x_1, x_2, x_3 in the highest terms of $f(x_1, x_2, x_3)$, $g(x_1, x_2, x_3)$, $h(x_1, x_2, x_3)$, …, are distinct triples, and because the sum of the integers in each triple is either 0 or 8, the degree of these homogeneous polynomials in x_1, x_2, x_3.

PROBLEMS

In each of the following problems if n and f are as stated, find the polynomial in $E_1, \ldots, E_n$ and the coefficients of f to which f is equal identically in $x_1, \ldots, x_n$.

1. 3, $b\Sigma x_1^2x_2x_3 + c\Sigma x_1^2x_2^2$
2. 3, $b\Sigma x_1^3x_2 + c\Sigma x_1^2x_2x_3$
3. 3, $b\Sigma x_1^2x_2 + c\Sigma x_1^2x_2$
4. 3, $b\Sigma x_1^2x_2x_3 + c\Sigma x_1^2x_2$
5. 3, $b\Sigma x_1^2x_2x_3 + c\Sigma x_1^2x_2^2x_3$
6. 3, $b\Sigma x_1^2x_2^2x_3 + c\Sigma x_1^2x_2^2$
7. 3, $b\Sigma x_1^3x_2^2 + c\Sigma x_1^2x_2x_3 + d\Sigma x_1^2x_2$
8. 3, $b\Sigma x_1^3x_2x_3 + c\Sigma x_1^2x_2^2 + d\Sigma x_1^2x_2$
9. 3, $b\Sigma x_1^3x_2^3 + c\Sigma x_1^3x_2^2x_3$
10. 3, $b\Sigma x_1^3x_2^2x_3 + c\Sigma x_1^2x_2^2x_3^2$
11. 4, $b\Sigma x_1^2x_2^2x_3 + c\Sigma x_1^2x_2x_3$
12. 4, $b\Sigma x_1^2x_2^2 + c\Sigma x_1^2x_2x_3x_4$
13. 4, $b\Sigma x_1^2x_2^2x_3x_4 + c\Sigma x_1^3x_2^2x_3$
14. 4, $b\Sigma x_1^3x_2x_3x_4 + c\Sigma x_1^4x_2x_3$
15. 4, $b\Sigma x_1^4x_2x_3 + c\Sigma x_1^3x_2^2x_3$
16. 4, $b\Sigma x_1^4x_2^2 + c\Sigma x_1^3x_2^2x_3$

No new ideas are involved in the proof that if n is a positive integer and if f is a symmetric homogeneous polynomial in $x_1, \ldots, x_n$, then there is a polynomial F in $E_1, \ldots, E_n$ and the coefficients of f with integral coefficients such that f equals F identically in $x_1, \ldots, x_n$. If the degree of f in $x_1, \ldots, x_n$ is zero, then the polynomial F is precisely the polynomial f. Therefore it is assumed in the following proof that the degree k of f in $x_1, \ldots, x_n$ is a positive integer. If $k_1, \ldots, k_n$ are integers, each of which is positive or zero and if $k_1 + \cdots + k_n = k$, then there is only a finite number of possible values for each of $k_1, \ldots, k_n$. Therefore the list of symbols which is obtained from $(k_1, \ldots, k_n)$ by giving such values to $k_1, \ldots, k_n$ in all possible ways is of finite length.

If the highest term in f is

then the coefficient b and the exponents $h_1, \cdots, h_n$ in this highest term are used to construct the expression

$$(65) \qquad bE_1^{h_1-h_2}E_2^{h_2-h_3}\cdots E_{n-1}^{h_{n-1}-h_n}E_n^{h_n}.$$

By (31), (65) is a symmetric homogeneous polynomial of degree $h_1+\cdots+h_n$ in $x_1, \cdots, x_n$. If f_1 is defined by

$$(66) \quad f_1(x_1, \cdots, x_n) \equiv f(x_1, \cdots, x_n) - bE_1^{h_1-h_2}E_2^{h_2-h_3}\cdots E_{n-1}^{h_{n-1}-h_n}E_n^{h_n},$$

then f_1 is a symmetric homogeneous polynomial in $x_1, \cdots, x_n$ which is either identically zero or of degree $h_1+\cdots+h_n$. By (31) and lemma 1, the highest term in (65) is, in fact, the highest term (64) in f. Therefore, by (66), a term which has $x_1^{h_1}\cdots x_n^{h_n}$ as a factor does not appear in f_1. Therefore f is higher than f_1.

If f_1 is identically zero, then, by (66), the required polynomial in $E_1, \cdots, E_n$ and the coefficients of f is (65). If f_1 is not identically zero, then the argument which has been applied to f is repeated on f_1. Thus, expressions for f_1, which are analogous to (64), (65), (66) for f, are written. The new function f_2 is a symmetric homogeneous polynomial in $x_1, \cdots, x_n$, which is either identically zero or of degree $h_1+\cdots+h_n$ in $x_1, \cdots, x_n$. In the former case the required polynomial has been found. In the latter case the argument is repeated on f_2.

It will now be proved that in the sequence $f, f_1, f_2, \cdots$, to which this process leads, there is a function which is identically zero and hence that this process terminates. First, each of the exponents $h_1, \cdots, h_n$ in the highest term (64) of f is a positive integer or zero, and the sum of these exponents is the degree k of f. Thus the symbol $(h_1, \cdots, h_n)$ is one of the symbols in the list which was described earlier. If f_1 is not identically zero, the same statements are true of the exponents in the highest term of f_1. They are also true for each f_i in the sequence to which this process leads, if f_i is not identically zero. Also, if f_i and f_j are two such functions in this sequence, and if $i \neq j$, then the symbol for f_i is different from the symbol for f_j. All these different symbols appear in a list of finite length. Therefore in the sequence $f, f_1, f_2, \cdots$ there is a function which is identically zero. The process terminates with this function Therefore there is a finite set of identities of which (66) is the first. If each of these is substituted in turn in the preceding identity, the final identity obtained by substitution in (66) yields the desired polynomial F

It is to be noted especially that the degree of (65) in $E_1 \quad E_n$ is h_1 and that h_1 is the exponent on x_1 in the highest term (64) of f Again the degree in $E_1 \quad E_n$ of the term which is subtracted from f_1 to give f_2 is the exponent on x_1 in the highest term of f_1 Similar statements hold for the remaining identities Therefore the degree in $E_1 \quad E_n$ of the desired polynomial F is the largest of these exponents on x_1 Now h_1 is the largest of these exponents because each polynomial in the sequence $f \; f_1 \; f_2$ is higher than the polynomials which follow it Therefore h_1 is the degree of F in $E_1 \quad E_n$ For example in (50) $h_1 = 4$ Also $g(x_1 \; x_2 \; x_3)$ in (58) is $f_1(x_1 \quad x_n)$ in this case The exponent on x_1 in the highest term (59) of g is 3 and the degree of (60) in $E_1 \; E_2 \; E_3$ is 3 The degree of the right hand side of (63) in $E_1 \; E_2 \; E_3$ is 4

It is also to be noted that (65) is a polynomial in $E_1 \quad E_n$ and the coefficients of f with integral coefficients By (66) the coefficients of f are combined only by addition and subtraction to give the coefficients of f_1 Therefore the term which is subtracted from f_1 to give f_2 is a polynomial in $E_1 \quad E_n$ and the coefficients of f with integral coefficients An analogous statement holds of all the terms subtracted Since F is the sum of these terms it follows that F is a polynomial in $E_1 \quad E_n$ and the coefficients of f with integral coefficients This completes the proof of theorem 2 *the fundamental theorem on symmetric polynomials*

THEOREM 2 *If n is a positive integer if $x_1 \quad x_n$ are independent variables if the elementary symmetric functions $E_1 \quad E_n$ of these variables are defined by (31) and if f is a symmetric polynomial in $x_1 \quad x_n$ then there is a polynomial F in $E_1 \quad E_n$ and the coefficients of f with intergral coefficients such that in accordance with the definition which follows (49) f equals F identically in $x_1 \quad x_n$ If the degree of f in x_1 is h_1 then the degree of F in $E_1 \quad E_n$ is h_1*

Other facts about symmetric polynomials are given in the references cited at the end of the book

3 Resultants Discriminants The definition and importance of resultants will now be illustrated with $f(x)$ and $g(x)$ defined by

(67) $$f(x) = a_0x^3 + a_1x^2 + a_2x + a_3 \qquad a_0 \neq 0$$

(68) $$g(x) \equiv b_0x^2 + b_1x + b_2 \qquad b_0 \neq 0$$

If r_1, r_2, r_3 are the roots of $f(x) = 0$, and if s_1, s_2 are the roots of $g(x) = 0$, then by theorem 15 of chapter 3

$$f(x) \equiv a_0(x - r_1)(x - r_2)(x - r_3), \tag{69}$$

$$g(x) \equiv b_0(x - s_1)(x - s_2). \tag{70}$$

Now r_1, r_2, r_3 are also the roots of the equation obtained by dividing $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$ by a_0. The resulting equation is given the notation

$$x^3 + a'_1x^2 + a'_2x + a'_3 = 0. \tag{71}$$

Therefore

$$a'_1 = \frac{a_1}{a_0}, \quad a'_2 = \frac{a_2}{a_0}, \quad a'_3 = \frac{a_3}{a_0}. \tag{72}$$

If the elementary symmetric functions of r_1, r_2, r_3 are designated by A_1, A_2, A_3, then, by definition,

$$A_1 = r_1 + r_2 + r_3, \quad A_2 = r_1r_2 + r_1r_3 + r_2r_3, \quad A_3 = r_1r_2r_3. \tag{73}$$

Therefore, by theorem 1 with $n = 3$,

$$A_1 = -a'_1, \quad A_2 = a'_2, \quad A_3 = -a'_3. \tag{74}$$

The product $g(r_1)g(r_2)g(r_3)$ is a symmetric function of the roots r_1, r_2, r_3 of (71). By (68),

$$g(r_1)g(r_2)g(r_3) = (b_0r_1^2 + b_1r_1 + b_2)(b_0r_2^2 + b_1r_2 + b_2)(b_0r_3^2 + b_1r_3 + b_2). \tag{75}$$

If $n = 3$ in theorem 2, and if x_1, x_2, x_3 are replaced respectively by r_1, r_2, r_3, then E_1, E_2, E_3 are replaced by A_1, A_2, A_3. Therefore, by theorem 2, there is a polynomial F in A_1, A_2, A_3 and the coefficients b_1, b_2, b_3, such that F equals $g(r_1)g(r_2)g(r_3)$ identically in r_1, r_2, r_3. The degree of F in A_1, A_2, A_3 is 2, because, by (75), the degree of $g(r_1)g(r_2)g(r_3)$ in r_1 is 2. Therefore, by (74) and (72), F is a polynomial of degree 2 in a_1/a_0, a_2/a_0, a_3/a_0. If this polynomial is multiplied by a_0^2, the result is an expression in which a_0 does not appear in a denominator. The result is indeed a homogeneous polynomial of degree 2 in a_0, a_1, a_2, a_3. It is designated by $R(f, g)$ and is called the resultant of f and g. Thus $R(f, g)$ is a homogeneous polynomial of degree 2 in a_0, a_1, a_2, a_3, which is also a polynomial in b_0, b_1, b_2, such that

$$R(f, g) = a_0^2g(r_1)g(r_2)g(r_3). \tag{76}$$

By (76) and (70),

$$(77)\quad R(f, g) = a_0^2b_0^3(r_1 - s_1)(r_1 - s_2)(r_2 - s_1)(r_2 - s_2)(r_3 - s_1)(r_3 - s_2)$$

If the symbol $\prod_{i,j}$ designates the product obtained when i takes the values 1, 2, 3 and independently j takes the values 1, 2, then

$$(78)\qquad R(f, g) = a_0^2b_0^3 \prod_{i,j} (r_i - s_j)$$

By (77), $R(f, g) = 0$ if and only if one of the roots r_1, r_2, r_3 equals one of the roots s_1, s_2. Therefore $R(f, g) = 0$ is a necessary and sufficient condition that $f(x) = 0$ and $g(x) = 0$ have a common root. This fact is also expressed by the statement that x has been eliminated between $f(x) = 0$ and $g(x) = 0$. This discussion is an illustration of a simple part of the theory of elimination.

It is to be noted especially that f and g are used in different ways in the preceding discussion. Therefore, if this method were applied to g and f instead of to f and g, the resultant $R(g, f)$ of g and f would be obtained. Thus $R(g, f)$ is a homogeneous polynomial of degree 3 in b_0, b_1, b_2, which is also a polynomial in a_0, a_1, a_2, a_3, such that

$$(79)\qquad R(g, f) = b_0^3f(s_1)f(s_2)$$

Also

$$(80)\quad R(g, f) = b_0^3a_0^2(s_1 - r_1)(s_1 - r_2)(s_1 - r_3)(s_2 - r_1)(s_2 - r_2)(s_2 - r_3),$$

$$(81)\quad R(g, f) = b_0^3a_0^2 \prod_{i,j} (s_j - r_i)$$

Now, by (80),

$$(82)\quad R(g, f) = (-1)^6b_0^3a_0^2(r_1 - s_1)(r_2 - s_1)(r_3 - s_1)(r_1 - s_2)(r_2 - s_2)(r_3 - s_2)$$

Therefore by (77), for the particular polynomials (67) and (68)

$$(83)\qquad R(g, f) = (-1)^6R(f, g)$$

No new ideas are involved in a discussion of the resultant of the polynomials $f(x)$ and $g(x)$ defined by

$$(84)\quad f(x) \equiv a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m, \quad a_0 \neq 0,$$

$$(85)\quad g(x) \equiv b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n, \quad b_0 \neq 0$$

By the fundamental theorem of algebra, there are m roots $r_1, \cdots, r_m$ of $f(x) = 0$ and there are n roots $s_1, \cdots, s_n$ of $g(x) = 0$. The product $g(r_1) \cdots g(r_m)$ is a symmetric polynomial in $r_1, \cdots, r_m$. By theorem 2 there is a polynomial F in the elementary symmetric functions $A_1, \cdots, A_m$ of $r_1, \cdots, r_m$ and the coefficients $b_0, \cdots, b_n$, such that F equals $g(r_1) \cdots g(r_m)$ identically in $r_1, \cdots, r_m$. Now F is of degree n in $A_1, \cdots, A_m$ because $g(r_1) \cdots g(r_m)$ is of degree n in r_1. Also $A_i = (-1)^i a_i/a_0$ for $i = 1, \cdots, m$. If these substitutions are made for $A_1, \cdots, A_m$ in F, and if the result is multiplied by $a_0{}^n$, a homogeneous polynomial of degree n in $a_0, \cdots, a_m$ is obtained. It is designated by $R(f, g)$ and is called *the resultant of f and g*. Thus $R(f, g)$ is a homogeneous polynomial of degree n in $a_0, \cdots, a_m$, which is also a polynomial in $b_0, \cdots, b_n$, such that

$$R(f, g) = a_0{}^n g(r_1) \cdots g(r_m). \tag{86}$$

By theorem 15 of chapter 3, $f(x) \equiv a_0(x - r_1) \cdots (x - r_m)$ and $g(x) \equiv b_0(x - s_1) \cdots (x - s_n)$. If the symbol $\prod_{i,j}$ designates the product obtained when i takes the values $1, \cdots, m$ and j independently takes the values $1, \cdots, n$, then

$$R(f, g) = a_0{}^n b_0{}^m \prod_{i,j} (r_i - s_j). \tag{87}$$

Therefore *$R(f, g) = 0$ if and only if the equations $f(x) = 0$ and $g(x) = 0$ have a common root.*

Similarly, the resultant $R(g, f)$ of g and f is a homogeneous polynomial of degree m in $b_0, \cdots, b_n$, which is also a polynomial in $a_0, \cdots, a_m$, such that $R(g, f) = b_0{}^m f(s_1) \cdots f(s_n)$. Also, $R(g, f) = b_0{}^m a_0{}^n \prod_{i,j} (s_j - r_i)$. Therefore

$$R(g, f) = (-1)^{mn} R(f, g). \tag{88}$$

This completes the proof of theorem 3.

THEOREM 3. *If $f(x)$ and $g(x)$ are the polynomials (84) and (85), and if the roots of $f(x) = 0$ are designated by $r_1, \cdots, r_m$ and the roots of $g(x) = 0$ by $s_1, \cdots, s_n$, then $g(r_1) \cdots g(r_m)$ is a symmetric polynomial in $r_1, \cdots, r_m$. There is a homogeneous polynomial of degree n in $a_0, \cdots, a_m$, which is also a polynomial in $b_0, \cdots, b_n$, and which is designated by $R(f, g)$, such that $R(f, g) = a_0{}^n g(r_1) \cdots g(r_m)$. $R(f, g)$ is called the resultant of f and g. $R(f, g) = 0$ is a*

necessary and sufficient condition that $f(x) = 0$ and $g(x) = 0$ have a common root

PROBLEMS

If f and g are as stated in each of the following problems find $R(f\ g)$

1 $a_0x + a_1 \quad b_0x + b_1$
2 $a_0x + a_1 \quad b_0x^2 + b_1x + b_2$
3 $a_0x^2 + a_1x + a_2 \quad b_0x^2 + b_1x + b_2$
4. $a_0x^3 + a_1x^2 + a_2x + a_3 \quad b_0x + b_1$
5 $a_0x^3 + a_1x^2 + a_2x + a_3 \quad b_0x^2 + b_1x + b_2$
6 $a_0x^3 + a_1x^2 + a_2x + a_3 \quad b_0x^3 + b_1x^2 + b_2x + b_3$
7 $a_0x^4 + a\,x^3 + a_2x^2 + a_3x + a_4 \quad b_0x + b_1$
8 $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \quad b_0x^2 + b_1x + b_2$
9 $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \quad b_0x^3 + b_1x^2 + b_2x + b_3$
10 $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \quad b_0x^4 + b_1x^3 + b_2x^2 + b_3x + b_4$

The discriminant of the general cubic equation was defined in theorem 4 of chapter 2

The discriminant of the general quartic equation was defined in theorem 8 of that chapter The discriminant of the general polynomial equation $f(x) = 0$ of degree n will now be discussed If $f(x)$ is given the notation (84) and $r_1 \quad r_m$ are the roots of $f(x) = 0$ then the product $(r_1 - r_2)^2 \quad (r_1 - r_m)^2(r_2 - r_3)^2 \quad (r_2 - r_m)^2 \quad (r_{m-1} - r_m)^2$ is a symmetric polynomial in $r_1 \quad r_m$ If the symbol $\prod_{<j}$ designates the product obtained when i takes the values $1 \quad m - 1$ and independently j takes the values $2 \quad m$ which are such that $i < j$ then the preceding symmetric polynomial in $r_1 \quad r_m$ can be written $\prod_{i<j} (r - r_j)^2$ Therefore by theorem 2 there is a polynomial F in $A_1 \quad A_m$ such that F equals $\prod_{<j} (r - r)^2$ identically in $r_1 \quad r_m$ Now $\prod_{<} (r - r_j)^2$ is of degree $2(m - 1)$ in r_1 Also $A = (-1)^i a / a_0$ Therefore $a_0^{2m-2}F$ is a homogeneous polynomial of degree $2m - 2$ in $a_0 \quad a_m$ This polynomial is called *the discriminant of* f and is designated by δ Therefore δ is a homogeneous polynomial of degree $2m - 2$ in $a_0 \quad a_m$ such that

$$\delta = a_0^{2m-2} \prod_{<} (r_i - r_j)^2 \tag{89}$$

Now by (89) $\delta = 0$ *is a necessary and sufficient condition that* $f(x) = 0$ *shall have a multiple root*

By theorems 17 and 18 of chapter 3, $f(x) = 0$ has a multiple root if and only if $f(x) = 0$ and $f'(x) = 0$ have a common root, and hence, by theorem 3, if and only if $R(f, f') = 0$. It will now be proved that

$$R(f, f') = (-1)^{m(m-1)/2} a_0 \delta. \tag{90}$$

By theorem 15 of chapter 3, $f(x) \equiv a_0(x - r_1) \cdots (x - r_m)$. Therefore, by the rule for differentiating a product,

$$\begin{aligned}
f'(r_1) &= a_0(r_1 - r_2)(r_1 - r_3) \cdots (r_1 - r_m),\\
f'(r_2) &= a_0(r_2 - r_1)(r_2 - r_3) \cdots (r_2 - r_m),\\
&\cdot \qquad\qquad \cdot\\
&\cdot \qquad\qquad \cdot\\
&\cdot \qquad\qquad \cdot\\
f'(r_m) &= a_0(r_m - r_1)(r_m - r_2) \cdots (r_m - r_{m-1}).
\end{aligned}$$

Therefore, by (86), with g replaced by f' and n by $m - 1$,

$$\begin{aligned}
R(f, f') = a_0^{m-1} \cdot a_0^{m} \cdot (-1)^{m-1}(r_1 - r_2)^2 \cdots (r_1 - r_m)^2 \cdot \\
(-1)^{m-2}(r_2 - r_3)^2 \cdots (r_2 - r_m)^2 \cdots \\
(-1)^2(r_{m-2} - r_{m-1})^2(r_{m-2} - r_m)^2 \\
\cdot (-1)(r_{m-1} - r_m)^2.
\end{aligned}$$

Also $(-1)^{(m-1)+(m-2)+\cdots+2+1} = (-1)^{m(m-1)/2}$. Therefore

$$R(f, f') = a_0^{2m-1}(-1)^{m(m-1)/2} \prod_{i<j} (r_i - r_j)^2. \tag{91}$$

If (89) is used in (91), the result is (90).

This completes the proof of theorem 4.

Theorem 4. *If $f(x)$ is the polynomial (84) and if the roots of $f(x) = 0$ are designated by $r_1, \cdots, r_m$, then $\prod_{i<j} (r_i - r_j)^2$ is a symmetric polynomial in $r_1, \cdots, r_m$. There is a homogeneous polynomial of degree $2m - 2$ in $a_0, \cdots, a_m$, which is designated by δ, such that $\delta = a_0^{2m-2} \prod_{i<j} (r_i - r_j)^2$. δ is called the discriminant of $f(x)$. $\delta = 0$ is a necessary and sufficient condition that $f(x) = 0$ shall have a multiple root. Equation (90) relates δ and $R(f, f')$.*

If $m = 3$ and δ is found by the method explained in the proof of theorem 2 the express on which was given in theorem 4 of chapter 2 is obtained Another method of obtaining the homogeneous polynomial in $a_0 \quad a_m$ which is $R(f\ g)$ will be explained later This will give in particular a new method of obtaining $R(f\ f\ )$ and hence by (90) a new method of obtaining δ

It will now be proved that if $f(x)$ and $g(x)$ are defined by (67) and (68) then $f(x) = 0$ and $g(x) = 0$ have a common root if and only if

$$(92) \qquad \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a & a_2 & a_3 \\ b_0 & b_1 & b_2 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 \end{vmatrix} = 0$$

First it will be proved that if $f(x) = 0$ and $g(x) = 0$ have a common root then there is a polynomial $f_1(x)$ of degree 1 and a polynomial $g_1(x)$ of degree 2 such that

$$(93) \qquad f_1(x)f(x) = g_1(x)g(x)$$

If $f(x) = 0$ and $g(x) = 0$ have a common root then the notation for the roots can be chosen so that $r_1 = s_1$ Then (93) holds with $f_1(x)$ defined to be $b_0(x \quad s_2)$ and $g\ (x)$ defined to be $a_0(x - r_2)(x - r_3)$ Next it will be proved that if there are polynom als $f_1(x)$ and $g_1(x)$ of degrees 1 and 2 respectively such that (93) holds then $f(x) = 0$ and $g(x) = 0$ have a common root By (93) and (69) the product $(x - r_1)(x - r_2)(x - r_3)$ is a factor of the polynomial of degree 4 in x wh ch is obtained if the operations indicated in $g_1(x)g(x)$ are performed By the fundamental theorem of algebra and theorem 15 of chapter 3 $g_1(x)$ is a product of two linear factors Therefore one of the linear factors $x - r \quad x - r_2 \quad x - r_3$ is a factor of $g(x)$ Therefore $f(x) = 0$ and $g(x) = 0$ have a common root

If the notations

$$(94) \qquad f_1(x) \equiv c_0x + c_1 \qquad c_0 \neq 0$$

$$(95) \qquad g_1(x) = d_0x^2 + d_1x + d_2 \quad d_0 \neq 0$$

are used then (93) is equivalent by (67) and (68) to $(c_0x + c_1)(a_0x^3 + a_1x^2 + a_2x + a_3) = (d_0x^2 + d_1x + d_2)(b_0x^2 + b_1x + b_2)$ and hence to $(a_0c_0 - b_0d_0)x^4 + (a_1c_0 + a_0c_1 - b_1d_0 - b_0d_1)x^3 +$

$(a_2c_0 + a_1c_1 - b_2d_0 - b_1d_1 - b_0d_2)x^2 + (a_3c_0 + a_2c_1 - b_2d_1 - b_1d_2)x + (a_3c_1 - b_2d_2) \equiv 0$. Hence (93) is equivalent to

$$
(96) \qquad \begin{aligned}
a_0c_0 \qquad\qquad\quad - b_0d_0 \qquad\qquad\qquad\qquad &= 0,\\
a_1c_0 + a_0c_1 - b_1d_0 - b_0d_1 \qquad\qquad &= 0,\\
a_2c_0 + a_1c_1 - b_2d_0 - b_1d_1 - b_0d_2 &= 0,\\
a_3c_0 + a_2c_1 \qquad\qquad - b_2d_1 - b_1d_2 &= 0,\\
a_3c_1 \qquad\qquad\qquad\qquad - b_2d_2 &= 0.
\end{aligned}
$$

It has been proved, therefore, that $f(x) = 0$ and $g(x) = 0$ have a common root if and only if there are numbers c_0, c_1, d_0, d_1, d_2 such that $c_0 \neq 0$, $d_0 \neq 0$, and (96) hold. Now the five linear homogeneous equations

$$
(97) \qquad \begin{aligned}
a_0z_1 \qquad\qquad + b_0z_3 \qquad\qquad\qquad\qquad &= 0,\\
a_1z_1 + a_0z_2 + b_1z_3 + b_0z_4 \qquad\qquad &= 0,\\
a_2z_1 + a_1z_2 + b_2z_3 + b_1z_4 + b_0z_5 &= 0,\\
a_3z_1 + a_2z_2 \qquad\qquad + b_2z_4 + b_1z_5 &= 0,\\
a_3z_2 \qquad\qquad\qquad\qquad + b_2z_5 &= 0,
\end{aligned}
$$

have a solution which is not the zero solution if and only if

$$
(98) \qquad \begin{vmatrix}
a_0 & 0 & b_0 & 0 & 0\\
a_1 & a_0 & b_1 & b_0 & 0\\
a_2 & a_1 & b_2 & b_1 & b_0\\
a_3 & a_2 & 0 & b_2 & b_1\\
0 & a_3 & 0 & 0 & b_2
\end{vmatrix} = 0.
$$

Also, this determinant is zero if and only if the determinant which is obtained from it by interchanging rows and columns is zero. This proves the statement involving (92).

It is proved in the references cited at the end of this book that, if $f(x)$ and $g(x)$ are defined by (67) and (68), then $R(f, g)$ is the determinant on the left-hand side of (92).

If $f(x)$ and $g(x)$ are defined by (84) and (85), then the determinant B which is analogous to the determinant on the left-hand side of (92) is defined in the following manner. There are $m + n$ rows and $m + n$ columns in B. The first row of B consists of the $m + 1$ elements $a_0, a_1, \cdots, a_m$ followed by $n - 1$ zeros; the second row

of B consists of the elements $0\ a_0\ a_1 \quad a_m$ followed by $n - 2$ zeros finally the nth row of B cons sts of $n - 1$ zeros followed by $a_0\ a_1 \quad a_m$ The row of B which is numbered $n + 1$ con s sts of $b_0\ b_1 \quad b_n$ followed by $m - 1$ zeros the row numbered $n + 2$ consists of $0\ b_0\ b_1 \quad b_n$ followed by $m - 2$ zeros finally the ro v numbered $n + m$ consists of $m - 1$ zeros followed by $b_0\ b_1 \quad b_n$ No new ideas are involved in the proof that $f(x) \quad 0$ and $g(x) = 0$ have a common root if and only if there is a polynomial $f_1(x)$ of degree $n - 1$ and a polynom al $g_1(x)$ of degree $m - 1$ such that (93) holds Therefore $f(x) = 0$ and $g(x) = 0$ have a common root if and only if the set of $m + n$ l near homogeneous equations in $m + n$ var ables v hose coeffic ent determinant is B has a solut on which is not the zero solution and hence if and only if $B = 0$ In more advanced treatments it is proved that $R(f\ g) = B$

PROBLEMS

If f and g a e as stated n the p oblems on page 262 write $R(f\ g)$ as a determi nant B Expand B and dent fy the result with that obtained by the former method

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ANSWERS

Page 5

3. -1, $(1 + \sqrt{3}i)/2$, $(1 - \sqrt{3}i)/2$.

Page 8

1. $\sqrt{2}(\cos 135° + i \sin 135°)$, $1(\cos 240° + i \sin 240°)$, $1(\cos 270° + i \sin 270°)$. 3. $\sqrt{2}(\cos 285° + i \sin 285°)$. 5. ω.

Page 14

3. z_0, z_2, z_4 are cube roots of unity.

Page 18

1. $r = 1$, $\theta = 40°$, $160°$, $280°$; $r = 1$, $\theta = 54°$, $126°$, $198°$, $270°$, $342°$. 3. $r = 1$, $\theta = 36°$, $108°$, $180°$, $252°$, $324°$; $r = 1$, $\theta = 24°$, $96°$, $168°$, $240°$, $312°$. 5. $r = \sqrt[6]{2}$, $\theta = 33° 45'$, $123° 45'$, $213° 45'$, $303° 45'$; $r = 1$, $\theta = 60°$, $132°$, $204°$, $276°$, $348°$.

Page 20

1. $r = \sqrt[6]{2}$, $\theta = 45°, 165°, 285°$; $r = \sqrt[6]{2}$, $\theta = -45°, 75°, 195°$. 3. $r = 2$, $\theta = 0°, 120°, 240°$ for 8 and for its conjugate. 5. $r = 1$, $\theta = 20°, 140°, 260°$; $r = 1$, $\theta = -20°, 100°, 220°$.

Page 25

1. $1/2$, $-1 + 2\omega - (\omega^2/2)$, $-1 + 2\omega^2 - (\omega/2)$. 3. $7/3$, $2 + \omega - (2\omega^2/3)$, $2 + \omega^2 - (2\omega/3)$. 5. $(1/3) + \sqrt[3]{2} - (2\sqrt[3]{4}/3)$, $(1/3) + \sqrt[3]{2}\omega - (2\sqrt[3]{4}\omega^2/3)$, $(1/3) + \sqrt[3]{2}\omega^2 - (2\sqrt[3]{4}\omega/3)$. 7. $-2 + \sqrt[3]{3} + \sqrt[3]{1/3}$, $-2 + \sqrt[3]{3}\omega + \sqrt[3]{1/3}\omega^2$, $-2 + \sqrt[3]{3}\omega^2 + \sqrt[3]{1/3}\omega$. 9. $-4/3$, $-1 + (2\omega/3) - \omega^2$, $-1 + (2\omega^2/3) - \omega$. 11. $1 - \sqrt[3]{2} + (1/3\sqrt[3]{2})$, $1 - \sqrt[3]{2}\omega + (\omega^2/3\sqrt[3]{2})$, $1 - \sqrt[3]{2}\omega^2 + (\omega/3\sqrt[3]{2})$. 13. $2 + \sqrt[3]{(5 + \sqrt{29})/2} + \sqrt[3]{(5 - \sqrt{29})/2}$, $2 + \omega\sqrt[3]{(5 + \sqrt{29})/2} + \omega^2\sqrt[3]{(5 - \sqrt{29})/2}$, $2 + \omega^2\sqrt[3]{(5 + \sqrt{29})/2} + \omega\sqrt[3]{(5 - \sqrt{29})/2}$. 15. $-2 + \sqrt[3]{-2 + \sqrt{3}} + \sqrt[3]{-2 - \sqrt{3}}$, $-2 + \omega\sqrt[3]{-2 + \sqrt{3}} + \omega^2\sqrt[3]{-2 - \sqrt{3}}$, $-2 + \omega^2\sqrt[3]{-2 + \sqrt{3}} + \omega\sqrt[3]{-2 - \sqrt{3}}$.

ANSWERS

Page 5

3. -1, $(1 + \sqrt{3}i)/2$, $(1 - \sqrt{3}i)/2$.

Page 8

1. $\sqrt{2}(\cos\ 135° + i\ \sin\ 135°)$, $1(\cos\ 240° + i\ \sin\ 240°)$, $1(\cos\ 270° + i \sin 270°)$. 3. $\sqrt{2}(\cos 285° + i \sin 285°)$. 5. ω.

Page 14

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Page 18

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Page 20

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Page 25

1. $1/2$, $-1 + 2\omega - (\omega^2/2)$, $-1 + 2\omega^2 - (\omega/2)$. 3. $7/3$, $2 + \omega - (2\omega^2/3)$, $2 + \omega^2 - (2\omega/3)$. 5. $(1/3) + \sqrt[3]{2} - (2\sqrt[3]{4}/3)$, $(1/3) + \sqrt[3]{2}\omega - (2\sqrt[3]{4}\omega^2/3)$, $(1/3) + \sqrt[3]{2}\omega^2 - (2\sqrt[3]{4}\omega/3)$. 7. $-2 + \sqrt[3]{3} + \sqrt[3]{1/3}$, $-2 + \sqrt[3]{3}\omega + \sqrt[3]{1/3}\omega^2$, $-2 + \sqrt[3]{3}\omega^2 + \sqrt[3]{1/3}\omega$. 9. $-4/3$, $-1 + (2\omega/3) - \omega^2$, $-1 + (2\omega^2/3) - \omega$. 11. $1 - \sqrt[3]{2} + (1/3\sqrt[3]{2})$, $1 - \sqrt[3]{2}\omega + (\omega^2/3\sqrt[3]{2})$, $1 - \sqrt[3]{2}\omega^2 + (\omega/3\sqrt[3]{2})$. 13. $2 + \sqrt[3]{(5 + \sqrt{29})/2} + \sqrt[3]{(5 - \sqrt{29})/2}$, $2 + \omega\sqrt[3]{(5 + \sqrt{29})/2} + \omega^2\sqrt[3]{(5 - \sqrt{29})/2}$, $2 + \omega^2\sqrt[3]{(5 + \sqrt{29})/2} + \omega\sqrt[3]{(5 - \sqrt{29})/2}$. 15. $-2 + \sqrt[3]{-2 + \sqrt{3}} + \sqrt[3]{-2 - \sqrt{3}}$, $-2 + \omega\sqrt[3]{-2 + \sqrt{3}} + \omega^2\sqrt[3]{-2 - \sqrt{3}}$, $-2 + \omega^2\sqrt[3]{-2 + \sqrt{3}} + \omega\sqrt[3]{-2 - \sqrt{3}}$.

Page 28

1 -1.206 -3.5321 -2.34730 3 1.3302 -3.1291 -1.20164 5 2.8608 -1.1149 1.2541 7 0.3221 -2.4912 -0.83092 9 2.83757 0.8928 2.26959 11 2.673 -0.4697 0.79664 13 -0.7330 -3.5257 -1.74135 15 0.8130 -2.1764 -0.13657

Page 32

1 $\Delta = 0$ 3 $\Delta > 0$ 5 $\Delta = 0$ 7 $\Delta < 0$ 9 $\Delta < 0$ 11 $\Delta < 0$ 13 $\Delta < 0$ 15 $\Delta = 0$

Page 36

1 $-1 \pm \sqrt{2}$ $1 \pm \sqrt{2}i$ 3 1 -2 $(1/2) \pm (\sqrt{3}/2)i$ 5 ω ω^2, $(1 \pm \sqrt{21})/2$ 7 $1 \pm i\sqrt{3}$ -1 -1 9 $(-1 \pm \sqrt{5})/2$ $(1 \pm i\sqrt{7})/2$ 11 $(-3 \pm i\sqrt{3})/2$ $(3 \pm \sqrt{17})/2$ 13 $(3 \pm \sqrt{5})/2$ $(-3 \pm \sqrt{17})/2$ 15 $3 \pm 4\sqrt{2}$ $3 \pm \sqrt{7}$ 17 $(-1 + \sqrt{2} \pm \sqrt{-29 + 14\sqrt{2}})/4$ $(-1 - \sqrt{2} \pm \sqrt{-29 - 14\sqrt{2}})/4$ 19 $(6 + 6\sqrt{11/3} \pm \sqrt{-34 + \sqrt{132}})/24$, $(6 - 6\sqrt{11/3} \pm \sqrt{-34 - \sqrt{132}})/24$

Page 42

1 $\delta > 0$ 3 $\delta < 0$ 5 $\delta > 0$ 7 $\delta = 0$ 9 $\delta < 0$ 11 $\delta = 0$

Page 46

1 1 3 -2 3 3 5 -2 1 5 1 -4 7 2 -3 9 1 2 3 -4 11 None 13 ± 1 4 15 ± 2 -5 7 17 2 6 19 None

Page 52

1 $r = -1$ 3 $r = -31$ 5 $r = 304$ 7 $r = 71$ 9 $r = 29$ 11 $r = 0$ 13 $r = 0$ 15 $r = 16.618$ 17 $r = -8.849$

Page 55

1 2 -3 3 ± 3 -2 -5 5 None 7 4 -2 9 5 -2 11 None 13 ± 2 ± 3 1 15 5

Page 61

1 0 -4 3 $1 + \sqrt{5}$ -8 5 $1 + \sqrt[3]{11}$ -3 7 $1 + \sqrt[3]{2}$ -2 9 $1 + \sqrt[4]{7}$ -2 11 8 $-1 - \sqrt[4]{7}$ 13 $5/2$ -2 15 4 0

Page 66

1 -3 4 $1/2$ $2/3$ 3 $3/4$ $-2/5$ 5 $4/3$ $-1/2$ 7 $2/9$ -1 3 4 9 None 11 -6 -1 2 13 2 $1/3$ $-1/2$ 15 None

Pages 70, 71

1. 1, double. 3. 2, triple. 5. None. 7. 2, double. 9. 3, triple. 11. None. 13. -1, 3, each double. 15. None.

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These answers list the roots of $f(x) = 0$. 1. $-2, -2, 3, 3$ 3. $-3, -3, -7, 4$. 5. $2, -3, \pm 1$. 7. $1, 1, 1, 2, 2$. 9. $2, 2, 2, 3, 3$. 11. $1, 1, 2, 2, -3$. 13. $1, 1, -2, -2, -3, -3$. 15. $1, 1, 1, 1, -3, -3$.

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1. $3, (-1 \pm i\sqrt{7})/2$ 3. $-1, 2, 3, (-3 \pm i\sqrt{7})/2$. 5. $1 \pm i\sqrt{2}$, each double. 7. $(1 \pm i\sqrt{15})/2$, each triple. 9. 1, triple; $(1 \pm i\sqrt{7})/2$, each double 11. $-2, -1$, each double; 2 13. $(3 \pm i\sqrt{7})/2$, each double.

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These answers list f_1, f_2, f_3 for problems 1, 3, 5, 9, 11 and f_1, f_2, f_3, f_4 for problems 7, 13, 15. 1. $3(x^2 - 2x - 5)$, $12(3x + 1)$, $-38 \cdot 12$. 3. $3(x^2 + 2x + 2)$, $6(-x + 7)$, $-6 \cdot 65$. 5. $3(x^2 + 8x + 4)$, $9(8x + 9)$, $3 \cdot 2151$. 7. $4(x^3 + 3x^2 - 2x + 1)$, $4(5x^2 - 5x + 2)$, $4(-8x + 3)$, $-4 \cdot 53$. 9. $3(x^2 - 4x)$, $6(4x - 1)$, 24. 11. $3(x^2 - 1)$, $3(2x - 5)$, -63 13. $4(x^3 + 6x^2 - 1)$, $12(4x^2 + x - 1)$, $12(19x - 7)$, $-32 \cdot 12$. 15. $2(2x^3 + 3x^2 + 4x + 1)$, $-2(5x^2 + 2x + 7)$, $-8(2x - 13)$, $8 \cdot 925$.

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These answers list consecutive integers between which the real roots lie. 1. $-3, -2; 0, 1; 5, 6$. 3. $1, 2$. 5. $-11, -10; -2, -1; 0, 1$. 7. $0, 1; -5, -4$. 9. $0, 1; 5, 6; -1, 0$. 11. $-3, -2$. 13. $-8, -7; -1, 0; 0, 0.37; 0.37, 1$. 15. None.

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These answers list consecutive integers between which the real roots lie. 1. $1, 2$. 3. $0, 1$. 5. $0, 1; 2, 3$. 7. $1, 2; 0, 1; -3, -2$. 9. $2, 3; -1, 0$.

Pages 112, 113

1. $x = 2$, $y = -3$, $z = -1$. 3. $y = 3x + 5$, $z = 2x - 1$. 5. $u/2 = v/(-3) = w/(-1)$. 7. $v = 3t - 1$, $s = 2t$. 9. Inconsistent. 11. $w = 0$, $u = 2v + 1$. 13. $v = 2s + 1$, $t = s - 2$.

Page 120

1. $6, 1, -4$. 3. $5, 3, 1$.

Page 122

1 $x = 2$ $y = -3$ $z = -1$ 3 Inconsistent 5 $v = 0$ $s = 0$ $t = 0$ 7 Inconsistent 9 $u = 2$ $v = -1$ $w = 3$ 11 Inconsistent

Pages 126, 127

1 $r = 3 = r_a$ $x = 2$ $y = -1$ $z = 11$ 3 $r = 2$ $r_a = 3$ inconsistent 5 $r = 2 = r_a$ infinitely many solutions 7 $r = 1$ $r_a = 2$ inconsistent 9 $r = 2 = r_a$, infinitely many solutions 11 $r = 3 = r_a$ $u = 0$ $s = 1$ $t = 3$

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1 $r = 4 = r_a$ $x = 2$ $y = -1$ $z = 1$ $w = 5$ 3 $r = 3$ $r_a = 4$ inconsistent 5 $r = 4 = r_a$ $x = 0 = y = z = w$ 7 $r = 3$ $r_a = 4$ inconsistent 9 $r = 4 = r_a$ $v = 0$ $s = -2$ $t = 1$ $w = -1$

Pages 139 140

3 $+a_{31}a_{42}a_{53}a_{14}a_{25}$ $-b_{31}b_{42}b_{23}b_{54}b_{15}$ $+a_{31}a_{42}a_{53}a_{34}a_{15}$ $-b_2\,b_{42}b_{13}b_{34}b_{55}$ $-a_{11}a_{42}a_{53}a_{34}a_{25}$ $+b_1\,b_{12}b_{23}b_{34}b_{55}$ 5 12 13 7 $+a_{21}a_{12}a_{53}a_{34}a_{75}a_{66}a_{47}$ $-b_2\,b_{72}b_{53}b_{34}b_{15}b_{66}b_{47}$ $-a_{71}a_{12}a_{23}a_{64}a_{45}a_{66}a_{37}$ $+b_7\,b_{42}b_{33}b_{54}b_{15}b_{66}b_{27}$ $-a_5\,a_{42}a_{63}a_{74}a_{55}a_{16}a_{37}$ $+b_{61}b_{52}b_{33}b_{74}b_{45}b_{26}b_{27}$

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3 $+a_{41}a_{32}a_{13}a_{54}a_{25}$ $+b_2\,b_{52}b_{23}b_{14}b_{45}$ $+a_{41}a_{32}a_{23}a_{14}a_{55}$ $+b_{41}b_{32}b_{23}b_{14}b_{55}$ $-a_5\,a_{42}a_{13}a_{24}a_{35}$ $b_3\,b_{42}b_{51}b_{24}b_{15}$ 5 $-a_3\,a_{12}a_{43}a_{74}a_{55}a_{26}a_{67}$ $-b_2\,b_{61}b_{13}b_{34}b_{65}b_{76}b_{47}$ $+a_{51}a_{72}a_{63}a_{14}a_{36}a_{26}a_{47}$ $+b_{41}b_{62}b_{53}b_{74}b_{16}b_{35}b_{27}$ $+a_4\,a_{73}a_{23}a_{14}a_{35}a_{66}a_{57}$ $+b_4\,b_{32}b_{63}b_{14}b_{75}b_{66}b_{27}$

Pages 152 153

1 -48 -75 87 -129 3 -112 63 -75 -53 5 942 7 651

Pages 161 162

1 -3565 3 -954 $-2\,954$ $-3\,954$ -954 954 5 254 208

Page 164

1 -29 35 3 -486 42 5 234 185 7 161 278.

Pages 186 187

1 $x = 1$ $y = 0$ $z = 2$ $w = -1$ 3 Inconsistent. 5 $v = 0$ $s = 1$ $t = -1$ $u = 2$ 7 $x = 0$ $y = 1$ $z = 0$ $u = -3$ $v = 2$ 9 Inconsistent

Pages 193, 194

1. $r = 3 = r_a$. 3. $r = 2$, $r_a = 3$. 5. $r = 3 = r_a$. 7. $r = 4$, $r_a = 5$. 9. $r = 3 = r_a$.

Pages 198, 199

3. $r = 4, r_a = 5$. 5. $r = 4, r_a = 5$. 7. $r = 3 = r_a$.

Page 201

Page 193, problem 1. $y = (13 - 3x)/9$, $z = (-35 + 6x)/9$, $u = -4/3$. Page 193, problem 5. $x = -(29 + 92y)/117$, $z = 7(1 + y)/18$, $t = (-17 + 3y)/26$. Page 194, problem 9. $s = -(13 + 13y + 51v)/12$, $t = (3 - y + 9v)/4$, $w = (5 - y + 27v)/3$. Page 199, problem 7. $y = (-31 + 42x - 59u)/17$, $z = (43 - 61x + 44u)/17$, $v = (21 - 46x + 29u)/17$.

Page 204

Page 193, problem 1. $f_4 = -f_1 + f_2 + 3f_3$ Page 193, problem 5. $f_2 = f_1 - 2f_3 + f_5$, $f_4 = -f_1 + 3f_3 + f_5$ Page 194, problem 9. $f_2 = f_1 - f_5 + f_6$, $f_3 = 2f_1 + f_5 + f_6$, $f_4 = f_1 + f_5 - f_6$. Page 199, problem 7. $f_2 = f_1 - f_4 - f_6$, $f_3 = f_1 + f_4 - 2f_6$, $f_5 = f_1 - f_4 + 2f_6$

Page 207

Page 193, problem 1. 0, 13/9, −35/9, −4/3; 1, 10/9, −29/9, −4/3; −1, 16/9, −41/9, −4/3 Page 193, problem 5 −29/117, 0, 7/18, −17/26; −112/117, 1, 28/9, −7/13; 7/13, −1, 0, −10/13. Page 194, problem 9. 0, − 13/12, 3/4, 0, 5/3; −1, 0, 1, 0, 2; 0, 16/3, 3, 1, 32/3 Page 199, problem 7. 0, −31/17, 43/17, 0, 21/17; 1, 11/17, −18/17, 0, −25/17; 1, −48/17, 26/17, 1, 4/17.

Pages 210, 211

1. $r = 3 = r_a$. 3. $r = 3$, $r_a = 4$. 5. $r = 4 = r_a$. 7. $r = 4$, $r_a = 5$. 9. $r = 2, r_a = 3$. 11. $r = 3 = r_a$.

Pages 212, 213

1. $r = 3$. 3. $r = 3$. 5. $r = 4$. 7. $r = 4$ 9. $r = 3$.

Page 218

1. $(3, 2, 8, -2), r = 2$ 3. $(-1, 11, 6, -7), r = 2$. 5. $(2, -1, 4, -3, 1), r = 3$. 7. $(2, 1, 3, 1, -2), r = 3$. 9. $(1, 1, -1, 2, -2), r = 3$.

Page 220

1. $r = 3$; $\zeta_4 = 3\zeta_1 - \zeta_2 - \zeta_3$. 3. $r = 2$; $2\zeta_3 = \zeta_1 + \zeta_2$, $2\zeta_4 = -3\zeta_1 + \zeta_2$. 5. $r = 3$; $4\zeta_4 = -\zeta_1 - \zeta_2 + 2\zeta_3$, $\zeta_5 = 3\zeta_1 - \zeta_2 - \zeta_3$. 7. $r = 4$; $\zeta_5 = -\zeta_1 - \zeta_2 + \zeta_3 + \zeta_4$, $\zeta_6 = -2\zeta_2 - \zeta_3 + \zeta_4$.

Pages 222, 223

1 $r = 2$ 3 $r = 2$ 5 $r = 3$ 7 $r = 4$ 9 $r = 3$

Page 226

3 $r = 2$ 5 $r = 4$ 7 $r = 3$

Pages 234 235

1 $r = 2$ $r_a = 3$. 3 $r = 2$ $r_a = 3$ 5 $r = 2 = r_a$

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